Low-frequency perturbation theory in eddy-current non-destructive evaluation

N. Harfield,[a] Y. Yoshida,[b] and J. R. Bowler
Department of Physics, University of Surrey, Guildford, Surrey GU2 5XH, England

(Received 22 April 1996; accepted for publication 18 June 1996)

A method is presented by which series solutions for the impedance change in an eddy-current test probe due to closed cracks in a non-magnetic, conducting half-space can be derived at low frequency. The series solution is applicable for flaws whose dimensions are much smaller than the electromagnetic skin-depth. The problem is formulated using an approach in which the flaw is represented by an equivalent distribution of current dipoles. The electric field scattered by the flaw is then written as an integral, over the flaw, of the product of the dipole density distribution and an appropriate Green's function. Terms in the series expansion for the dipole density are calculated by solving the integral equation at each order in the chosen small parameter, using perturbation theory and a dual integral equation method. The impedance change due to the crack is then calculated from the dipole distribution using the reciprocity theorem. Example solutions are given for semi-circular surface-breaking cracks and for long, uniformly deep surface-breaking cracks. Results are compared with other analytical solutions and the predictions of an independent numerical scheme, and very good agreement is observed. © *1996 American Institute of Physics.* [S0021-8979(96)08418-6]

I. INTRODUCTION

The detection and characterization of flaws in conducting material is important for the safe operation of many critical structures in, for example, the nuclear power and aerospace industries. Eddy-current inspection methods, in which the electric current is induced in a test-piece at a fixed frequency by an excitation coil, are commonly used to detect flaws such as cracks or inclusions by observing changes in impedance of the coil. The detection of a flaw is straightforward in comparison with its characterization, which requires detailed understanding of the relationship between observed signals and flaw geometry.

The literature concerning eddy-current inspection at low frequencies is relatively limited, probably due to the fact that flaw detection is far from optimum when the electromagnetic skin-depth, δ, is larger than the dimensions of the flaw. The skin-depth is given by

$$\delta=\left(\frac{2}{\omega\mu_0\sigma}\right)^{1/2}, \tag{1}$$

where $\omega=2\pi f$ is the angular frequency, μ_0 is the permeability of free space and σ is the conductivity. There are, however, both theoretical and experimental advantages in working with relatively large skin-depths. Theoretically, solutions for the electromagnetic fields may be sought in the form of a series expansion in the small parameter kl, where the wave number k is defined

$$k=\sqrt{i\omega\mu_0\sigma}=\frac{1+i}{\delta} \tag{2}$$

and l is a characteristic flaw dimension. At lowest order, the fields can be derived from a potential satisfying the Laplace equation, and higher order terms may be found using perturbation methods. This means that, for defects of simple geometry such as sub-surface spherical inclusions or surface-breaking hemispherical indentations, familiar analytical solutions can be used to determine the eddy-current distribution. The simple algebraic expressions obtained theoretically could be useful for probe calibration using a particular flaw geometry. Low-frequency operation is also required for significant field penetration into the conductor, which is particularly important in the detection of sub-surface flaws.

An early contribution to low-frequency eddy-current modelling is that of Burrows[1], who estimated the impedance change due to a spherical cavity in a conductor by modelling the cavity as a dipole source. Another approximate method was presented by Kincaid *et al.*[2] and Kincaid[3] who used the static form of Maxwell's equations to derive the scattered electric field due to an ellipsoidal void in a conductor for a uniformly applied electric field. The low-frequency limit in the case of a uniform applied field was also considered by Auld *et al.*[4] who adopted two different approaches. The flaw was modelled as a flat, hemi-ellipsoidal void breaking the conductor surface and, firstly, the scattered field was treated as that produced by an infinitesimal dipole located at the flaw centre with strength related to the size and aspect ratios of the ellipsoid. Secondly, the analogy between eddy-current flow in the low-frequency limit and fluid flow in an incompressible fluid was exploited.

A more thorough treatment of low-frequency eddy-current theory has been given by Nair and Rose.[5] Unlike the approximate methods described above, Nair and Rose considered the low-frequency asymptotics of a general formulation valid for arbitrary frequency. The first few terms of a low-frequency asymptotic expansion were derived for the electric fields induced by an external current distribution above a conducting half-space containing a flaw. From this

[a]Electronic mail: n.harfield@surrey.ac.uk
[b]Permanent address: Nuclear Engineering Research Laboratory, The University of Tokyo, 22-2 Shirakata-Shirane, Tokai-mura, Naka-gun, Ibaraki 319-11, Japan.

solution the flaw-induced impedance change was calculated. Example solutions were given for a number of sub-surface and surface-breaking defect geometries.

The solution method presented here is also developed from a general, rigorous formulation valid for arbitrary frequency. The crack is represented by an equivalent distribution of current dipoles and the electric field scattered by the flaw then expressed as an integral over the flaw of the scalar product of the dipole density distribution and an appropriate Green's function kernel. Terms in the series solution for the dipole density distrubution are calculated by solving the integral equation at each order in kl using a dual integral equation method.[6] The resulting series solution for the dipole density distribution is then used to calculate the impedance change via a relation derived using the reciprocity theorem.[7]

The structure of the paper is as follows. The problem formulation is described in Sec. II and the series solution is described in Sec. III. In Secs. IV and V specific problems are solved using the formulation of Sec. II, namely, the problems of a semi-circular surface-breaking crack (Sec. IV) and a long, surface-breaking crack of uniform depth (Sec. V). The results of these calculations are discussed in Sec. VI and compared with other analytical solutions (in the case of the semi-circular crack) and the results of a numerical scheme (in the case of the long crack). Concluding remarks are made in Sec. VII.

II. FORMULATION

Define an ideal crack as a crack which is closed but allows no passage of electric current. An ideal crack then produces a discontinuity in the tangential component of the scattered electric field which may be represented in terms of a surface distribution of current dipoles,[8] $\mathbf{p}=p\hat{n}$, where $\hat{n}$ is the unit vector normal to the crack:

$$\mathbf{E}_t^{+}-\mathbf{E}_t^{-}=-\frac{1}{\sigma}\nabla_t p(\mathbf{r}), \tag{3}$$

where the superscripts indicate limiting values as the crack is approached from one side or the other, and ∇_t is the differential operator tangential to the crack face. We adopt the convention that p is positive for dipoles directed towards the positive side of the layer. The dipole distribution effectively acts as the source of the scattered field. This means that the scattered field can be written in terms of an integral equation with a Green's function kernel which will be solved for $\mathbf{p}$. The impedance change due to the presence of the crack, ΔZ, will then be calculated by means of a relation derived from the reciprocity relationship[7]:

$$I^2\Delta Z=-\int_{S_0}\mathbf{E}^{(i)}\cdot\mathbf{p}\;dS, \tag{4}$$

where the superscript (i) indicates the incident (applied) field and S_0 is the open surface of the crack. The integral equation formulation lends itself to solution using perturbation methods and leads to an asymptotic series solution for both $\mathbf{p}$ and ΔZ. The field equations will be considered initially for an arbitrary current source and a conductor whose interface with air is defined by the plane $z=0$ and which contains a crack lying in the plane defined by $x=0$. The scattered electric field (denoted by the superscript (s)) can be expressed in integral form in the following way:

$$\mathbf{E}^{(s)}(\mathbf{r})=i\omega\mu_0\int_{S_0}\mathbf{G}(\mathbf{r},\mathbf{r}').\mathbf{p}(\mathbf{r}')dS', \tag{5}$$

where $\mathbf{G}(\mathbf{r},\mathbf{r}')$ is a half-space dyadic Green's function for a source in the conductor and the primed co-ordinates are source co-ordinates. The total electric field may, therefore, be written as the sum

$$\mathbf{E}(\mathbf{r})=\mathbf{E}^{(i)}(\mathbf{r})+i\omega\mu_0\int_{S_0}\mathbf{G}(\mathbf{r},\mathbf{r}').\mathbf{p}(\mathbf{r}')dS'. \tag{6}$$

The Green's function ensures that the electric field satisfies Maxwell's equations in the quasi-static limit and that the tangential components of the electric and magnetic fields are continuous at the air-conductor interface. An excellent discussion concerning the use of dyadic Green's functions in electromagnetic theory is given by Tai.[9] Equation (6) is appropriate for a three-dimensional system. In the specific examples which follow, the two-dimensional form of Eq. (6) will be solved by a perturbation method once suitable expressions for its components have been found. The incident electric field is known. The forms of the dyadic Green's function and the current dipole density distribution must be considered.

The dyadic Green's function appropriate to this problem may be written as follows[10]:

$$\mathbf{G}(\mathbf{r},\mathbf{r}')=\mathbf{G}_0(\mathbf{r},\mathbf{r}')+\mathbf{G}_i(\mathbf{r},\mathbf{r}')+\frac{1}{k^2}\nabla\times\hat{z}\nabla'\times\hat{z}V(\mathbf{r},\mathbf{r}'), \tag{7}$$

where $\mathbf{G}_0(\mathbf{r},\mathbf{r}')$ is the free-space dyadic Green's function, $\mathbf{G}_i(\mathbf{r},\mathbf{r}')$ represents the image source and the term containing $V(\mathbf{r},\mathbf{r}')$ deals with the effect of the air-conductor interface. The free-space dyadic Green's function, $\mathbf{G}_0(\mathbf{r},\mathbf{r}')$, is given by

$$\mathbf{G}_0(\mathbf{r},\mathbf{r}')=\left(\mathbf{I}+\frac{1}{k^2}\nabla\nabla\right)\phi(\mathbf{r},\mathbf{r}'), \tag{8}$$

where $\mathbf{I}=\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z}$ is the unit dyad and

$$\phi(\mathbf{r},\mathbf{r}')=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{9}$$

The Green's function representing the image source, $\mathbf{G}_i(\mathbf{r},\mathbf{r}')$, is given by

$$\mathbf{G}_i(\mathbf{r},\mathbf{r}')=\left(\mathbf{I}'-\frac{1}{k^2}\nabla\nabla'\right)\phi(\mathbf{r},\mathbf{r}''), \tag{10}$$

where $\mathbf{I}'=\hat{x}\hat{x}+\hat{y}\hat{y}-\hat{z}\hat{z}$ and, for a system in which the air-conductor interface occupies the plane $z=0$, $\mathbf{r}''=\mathbf{r}'-2\hat{z}\hat{z}'$ is the co-ordinate of the image of the point whose position vector is $\mathbf{r}'$. The two-dimensional Fourier representation of $V(\mathbf{r},\mathbf{r}')$ is

$$V(\mathbf{r},\mathbf{r}')=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{\kappa}-\frac{1}{\gamma}\right)\times e^{-\gamma|z+z'|+iu(x-x')+iv(y-y')}du\;dv, \tag{11}$$

where, taking roots with positive real part, $\kappa=\sqrt{u^2+v^2}$ and $\gamma=\sqrt{u^2+v^2-k^2}$ and u and v are Fourier space variables. From Eq. (11) it can be shown that a term of order k^2 is the lowest order term in $V(\mathbf{r},\mathbf{r}')$:

$$\frac{1}{\kappa}-\frac{1}{\gamma}=\frac{1}{\kappa}-\frac{1}{\kappa}\left(1-\frac{k^2}{\kappa^2}\right)^{-1/2}\sim\frac{1}{\kappa}-\frac{1}{\kappa}\left(1+\frac{k^2}{2\kappa^2}\right)=\frac{k^2}{2\kappa^3}.$$

While the contribution of $V(\mathbf{r},\mathbf{r}')$ must, in general, be considered, it will be shown that this term does not feature in the two-dimensional infinite crack problem (one of the examples considered later).

The first step in determining the current dipole density distribution, $\mathbf{p}$, is to rewrite Eq. (6) in terms of a magnetic vector potential, $\mathbf{A}$, which represents the scattered field:

$$\mathbf{E}(\mathbf{r})=\mathbf{E}^{(i)}(\mathbf{r})+\frac{1}{\mu_0\sigma}(\nabla\nabla+k^2\mathbf{I})\cdot\mathbf{A}(\mathbf{r}). \tag{12}$$

Imposing the Lorentz gauge condition on the divergence of $\mathbf{A}$ leads to the following equation which is valid in the conductor (but off the crack):

$$(\nabla^2+k^2)\mathbf{A}(\mathbf{r})=0. \tag{13}$$

The formal solution of Eq. (13) is[11]

$$\mathbf{A}(\mathbf{r})=\mu_0\int_{S_0}\mathbf{G}^{(\mathbf{A})}(\mathbf{r},\mathbf{r}').\mathbf{p}(\mathbf{r}')dS', \tag{14}$$

where $\mathbf{p}$ is a surface dipole density distribution. The dyadic Green's function for the vector potential is given by

$$\mathbf{G}^{(\mathbf{A})}(\mathbf{r},\mathbf{r}')=\mathbf{I}\phi(\mathbf{r},\mathbf{r}')+\mathbf{I}'\phi(\mathbf{r},\mathbf{r}'')+\frac{1}{k^2}\nabla\times\hat{z}\nabla'\times\hat{z}V(\mathbf{r},\mathbf{r}'). \tag{15}$$

Equation (12) becomes

$$\mathbf{E}(\mathbf{r})=\mathbf{E}^{(i)}(\mathbf{r})+\frac{1}{\sigma}(\nabla\nabla+k^2\mathbf{I})\int_{S_0}\mathbf{G}^{(\mathbf{A})}(\mathbf{r},\mathbf{r}').\mathbf{p}(\mathbf{r}')dS'. \tag{16}$$

By inserting Eq. (15) into Eq. (16) and taking care to exchange integral and differential operators only in such a way as to leave the equation unchanged, it may readily be verified that Eqs. (16) and (6) are equivalent.

The crack is assumed to act as a perfect barrier to the flow of eddy-currents which means that the normal component of the electric field at the surface of the crack is zero. Forming the scalar product of Eq. (16) with the unit vector normal to the crack face, $\hat{x}$, gives

$$J_x^{(i)}(\mathbf{r})+\left(\frac{\partial^2}{\partial x^2}+k^2\right)\int_{S_0}[\phi(\mathbf{r},\mathbf{r}')+\phi(\mathbf{r},\mathbf{r}'')]p(\mathbf{r}')dS'-\frac{\partial^2}{\partial y^2}\int_{S_0}V(\mathbf{r},\mathbf{r}')p(\mathbf{r}')dS'=0,\quad \mathbf{r}\in S_0, \tag{17}$$

where $J_x^{(i)}(\mathbf{r})=\sigma\mathbf{E}^{(i)}(\mathbf{r})\cdot\hat{x}$. The definition of the mixed boundary value problem is completed by noting that, in the remainder of the plane of the crack (the part not occupied by the crack itself) the boundary condition on the magnetic vector potential in the conductor is

$$\frac{\partial A_x(\mathbf{r})}{\partial x}=0,\quad \mathbf{r}\in S_1, \tag{18}$$

where S_1 denotes the part of the crack plane in the conductor not including the crack itself.

Now, the dipole density may be related to a jump in the normal gradient of the vector potential at the crack via Eq. (12). In finding this relationship it is observed that the vector potential of a vertical crack with its normal in the x-direction has only x- and y-components. The y-component arises solely from coupling between the x- and y-components of the vector potential at the surface of the conductor by the part of the Green's function $\mathbf{G}^{(\mathbf{A})}$, given by Eq. (15), which contains $V(\mathbf{r},\mathbf{r}')$. Since both the incident electric field and the field reflected at the air-conductor interface are continuous across the crack, it is found that the discontinuity in the tangential electric field across the crack depends only on the normal component of the vector potential:

$$\mathbf{E}_t(\mathbf{r})|_{x=0_+}-\mathbf{E}_t(\mathbf{r})|_{x=0_-}=\frac{1}{\mu_0\sigma}\nabla_t\left(\left.\frac{\partial A_x(\mathbf{r})}{\partial x}\right|_{x=0_+}-\left.\frac{\partial A_x(\mathbf{r})}{\partial x}\right|_{x=0_-}\right). \tag{19}$$

Comparing Eqs. (3) and (19) gives

$$p(\mathbf{r})=-\frac{1}{\mu_0}\left(\left.\frac{\partial A_x(\mathbf{r})}{\partial x}\right|_{x=0_+}-\left.\frac{\partial A_x(\mathbf{r})}{\partial x}\right|_{x=0_-}\right)=-\frac{2}{\mu_0}\left.\frac{\partial A_x(\mathbf{r})}{\partial x}\right|_{x=0_+}, \tag{20}$$

where the last step in Eq. (20) is a consequence of symmetry. Equations (17) and (20) together define the conditions which must be matched by the magnetic vector potential on the crack. Once the vector potential has been determined, the current dipole density on the crack is found from Eq. (20). The change in probe impedance due to the flaw can then be calculated using the relation given in Eq. (4).

III. SERIES SOLUTION

In the static limit, the incident electric field at the crack is zero and, consequently, the term in the series expansions for p of zero order in k, p_{k^0}, is also zero. Thus the series expansion for p starts at first order in k:

$$p(\mathbf{r})=p_{k^1}(\mathbf{r})+p_{k^2}(\mathbf{r})+\ \ldots \tag{21}$$

and, consequently,

$$\Delta Z = Z_{k^2} + Z_{k^3} + \ \ldots\ldots\ . \tag{22}$$

As mentioned in the introduction, solutions for $\partial A_x/\partial x$ (and hence for p_x) will be found by ordering the governing equations in terms of the small parameter kl so that, at each order, the individual terms in the series expansion for p are each a solution of Laplace's equation. This is the fundamental simplification afforded by the perturbation method of solution. At each order in kl we will define a mixed boundary value problem

$$\mathscr{J}(\mathbf{r}) + \frac{\partial^2 \mathscr{A}_x(\mathbf{r})}{\partial x^2} = 0, \quad \mathbf{r} \in \text{ crack domain} \tag{23}$$

$$\frac{\partial \mathscr{A}_x(\mathbf{r})}{\partial x} = 0, \quad \mathbf{r} \in \text{ crack plane excluding crack,} \tag{24}$$

where $\mathscr{J}$ and $\mathscr{A}_x$ are the terms of appropriate order in the series expansion representations of the current source $J_x^{(i)}$ and the x-component of the magnetic vector potential A_x. For simple geometries, this mixed boundary value problem lends itself to solution using a dual integral equation approach.

From Eq. (17) it can be seen that computation of the first two terms in the above series will, in general, be relatively straightforward compared with the calculation of higher order terms. The reason for this is that the term involving $V(\mathbf{r},\mathbf{r}')$ is of order k^2 at lowest order and not, therefore involved in the calculation of the first two terms in the series of Eqs. (21) and (22), simplifying the computation dramatically. In the example of the semi-circular, surface-breaking crack given in Sec. IV below, only the first two terms in these series are found. In the case of the long, surface-breaking crack of uniform depth treated in Sec. V, however, it is possible to proceed to higher order more easily because there is no y-dependence in the fields and the term containing $V(\mathbf{r},\mathbf{r}')$ in Eq. (17) vanishes for all orders of k.

IV. SEMI-CIRCULAR, SURFACE-BREAKING CRACK

We will now compute the first two terms in the series expansions for p and for ΔZ for a semi-circular, surface-breaking crack.

A. Dipole density

As mentioned above, the term involving $V(\mathbf{r},\mathbf{r}')$ in Eq. (17) does not enter the calculation of the first two terms in the series for p and ΔZ and, for these orders, we can write

$$J_x^{(i)} + \frac{\partial^2}{\partial x^2}\int_{S_0} [\phi(\mathbf{r},\mathbf{r}') + \phi(\mathbf{r},\mathbf{r}'')] p(\mathbf{r}') dS' = 0. \tag{25}$$

This means that, for these low orders, the half-disc crack can be transformed into a disc problem by extending the range of integration in Eq. (25) to embrace a circular region C_0. Similarly, Eq. (18) is then applied to a region C_1 consisting of the entire plane of the crack excluding C_0. To extend the region of integration in Eq. (25), the variable z' is replaced by $-z'$ for that part of the integration involving the image Green's function. The range of the functions $J_x^{(i)}$ and $p(\mathbf{r})$ is extended by assuming that they are even in z. This ensures that, over the half-range $z<0$, we obtain the same solution as we would from Eqs. (18) and (25). Equation (25) becomes

$$J_x^{(i)}(\mathbf{r}) + \frac{\partial^2}{\partial x^2}\int_{C_0} \phi(\mathbf{r},\mathbf{r}') p(\mathbf{r}') dS' = 0, \quad \mathbf{r} \in C_0, \tag{26}$$

that is,

$$J_x^{(i)}(\mathbf{r}) + \frac{\partial^2 A_x(\mathbf{r})}{\partial x^2} = 0, \quad \mathbf{r} \in C_0, \tag{27}$$

and Eq. (18) becomes

$$\frac{\partial A_x(\mathbf{r})}{\partial x} = 0, \quad \mathbf{r} \in C_1. \tag{28}$$

We can now order Eq. (26) explicitly. Write

$$J_x^{(i)} + \mathscr{K} p = 0 \tag{29}$$

where

$$\mathscr{K} = \left[\frac{\partial^2}{\partial x^2}\int_{C_0} dS' \, \phi(\mathbf{r},\mathbf{r}') \right]_{x=x'=0} \tag{30}$$

and, from Eq. (9),

$$\mathscr{K}_{k^0} = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{31}$$

Then p_{k^1} and p_{k^2} can be found by solving, respectively,

$$J_{k^1} + \mathscr{K}_{k^0} p_{k^1} = 0 \tag{32}$$

$$J_{k^2} + \mathscr{K}_{k^0} p_{k^2} = 0 \tag{33}$$

with Eq. (28).

For a uniform incident field directed normal to the crack face,

$$J_x^{(i)} = ikH_0 e^{ik|z|} = H_0\left[ik + (ik)^2\rho|\cos\theta| + \frac{(ik)^3}{2!}\rho^2\cos^2\theta + \cdots \right], \tag{34}$$

where we have used cylindrical polar co-ordinates, writing $z=\rho\cos\theta$, so

$$J_{k^1} = ikH_0 \tag{35}$$

$$J_{k^2} = -k^2 H_0 \rho |\cos\theta|. \tag{36}$$

Equations (32) and (33) may now readily be solved by noting the link with the magnetic vector potential through Eq. (27) and then solving for $\partial\mathscr{A}_x/\partial x$ using a dual integral equation method described in Ref. 6, especially Chap. 4. It can then be deduced from Eq. (20) that

$$p_{k^1}(\rho) = -4H_0 ik\sqrt{a^2-\rho^2}, \tag{37}$$

where a is the radius of the semi-disc shaped crack. The second order incident field is

$$J_{k^2}(\rho,\theta) = -k^2 H_0 \rho|\cos\theta| = -k^2 H_0\rho\left[\frac{2}{\pi} - \frac{4}{\pi}\sum_{\nu=1}^{\infty} \frac{(-1)^\nu \cos(2\nu\theta)}{4\nu^2-1} \right]. \tag{38}$$

Similarly expanding the second order dipole density as

$$p_{k^2}(\rho,\theta)=\sum_{n=0}^{\infty} p_{k^2}^{n}(\rho)\cos(n\theta) \tag{39}$$

and solving Eq. (33) and Eq. (28) gives, for the first term in this expression,

$$p_{k^2}^{0}=\frac{H_0k^2}{\pi}a\left[\sqrt{a^2-\rho^2}+\frac{\rho^2}{a}\ln\left(\frac{a+\sqrt{a^2-\rho^2}}{\rho}\right)\right]. \tag{40}$$

Only this first term in the expansion of Eq. (39) is required to calculate the contribution to the impedance since the first order incident field is independent of θ. The first two terms in the series expansion for the dipole density are thus given by Eqs. (32) and (33).

B. Impedance

The impedance change due to the crack will be calculated by means of Eq. (4) which is, in terms of the current source,

$$\Delta Z=-\frac{1}{\sigma I^2}\int_{S_0} J_x^{(i)}(\rho,\theta)p(\rho,\theta)dS \tag{41}$$

and, for the first two orders,

$$Z_{k^2}=-\frac{1}{\sigma I^2}\int_{S_0} J_{k^1}p_{k^1}dS, \tag{42}$$

$$Z_{k^3}=-\frac{1}{\sigma I^2}\int_{S_0}[J_{k^2}p_{k^1}+J_{k^1}p_{k^2}]dS. \tag{43}$$

Evaluation of the integral of Eq. (42) is straightforward, giving

$$Z_{k^2}=-\frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2\frac{4}{3}k^2a^3. \tag{44}$$

Calculation of the first term in Eq. (43) is also straightforward. In order to evaluate the second term in Eq. (43), the expression of Eq. (40) can be used since, for a uniform incident field, there is no contribution from higher order terms in Eq. (39). We obtain

$$Z_{k^3}=-\frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2 ik^3a^4. \tag{45}$$

This concludes the calculation of the first two terms in the low-frequency series expansion for the impedance change due to a semi-circular, surface-breaking flaw.

V. LONG, SURFACE-BREAKING CRACK

A. Dipole density

Consider a crack of infinite extent in the y-dimension with uniform depth d so that the crack surface is described by $x=0,-d\leqslant z\leqslant 0$. Both the crack geometry and the incident field are now independent of y which means that the dipole density is also independent of y. Integration by parts shows that the term containing V in Eq. (17) vanishes, leaving

$$J_x^{(i)}(\mathbf{r})+\left[\frac{\partial^2}{\partial x^2}+k^2\right]\int_{S_0}[\phi(\mathbf{r},\mathbf{r}')+\phi(\mathbf{r},\mathbf{r}'')]p(\mathbf{r}')dS'$$
$$=0,\quad \mathbf{r}\in S_0. \tag{46}$$

Equation (46), together with the condition of Eq. (18), defines the mixed boundary value problem in this two-dimensional case. For this geometry, image theory is applicable at all orders, enabling extension of the region of integration in Eq. (46) to include the image of the crack obtained on reflecting the cracked conductor in the interface plane. This extension is achieved, as for the semi-circular crack, by substituting $-z'$ for z' in the part of the integrand of Eq. (46) which involves the image Green's function. Eq. (46) becomes

$$J_x^{(i)}(\mathbf{r})+\left[\frac{\partial^2}{\partial x^2}+k^2\right]\int_{C_0}\phi(\mathbf{r},\mathbf{r}')p(\mathbf{r}')dS'=0,\quad \mathbf{r}\in C_0 \tag{47}$$

which, along with Eq. (28), defines the mixed boundary value problem for the long, surface-breaking crack of uniform depth in a half-space conductor.

For this long crack geometry, it is possible to integrate out the y' variable in the Green's function $\phi(\mathbf{r},\mathbf{r}')$ given by Eq. (9). The appropriate two-dimensional Green's function, representing an outgoing wave with a boundary at infinity, is given by

$$\int_{-\infty}^{\infty}\phi(|\mathbf{r}-\mathbf{r}'|)dy'=\frac{i}{4}H_0^{(1)}(kr),$$

where

$$|\mathbf{r}-\mathbf{r}'|=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2},$$
$$r=\sqrt{(x-x')^2+(z-z')^2}$$

and $H_0^{(1)}$ is the zeroth order Hankel function of the first kind. The Hankel function of the first kind is the correct Green's function in two dimensions for outgoing waves with a boundary at infinity. It exhibits a logarithmic singularity at the origin, as is required in the two-dimensional case. Equations (47) and (28) may now be written

$$\mu_0J_x^{(i)}(0,\ z)+\left[\left(\frac{\partial^2}{\partial x^2}+k^2\right)A_x(x,z)\right]_{x=x'=0}=0,\quad |z|<d, \tag{48}$$

where

$$A_x(x,z)=\mu_0\frac{i}{4}\int_{-d}^{d}H_0^{(1)}(kr)p(z')dz', \tag{49}$$

and

$$\frac{\partial A_x(x,z)}{\partial x}=0,\quad x=0,|z|>d \tag{50}$$

for the specific geometry in question. We will proceed to solve these equations for $\partial A_x/\partial x$ from which the dipole density can be deduced using Eq. (20).

The form of the series solutions obtained for the dipole density and impedance change is dictated by the form of the low-frequency asymptotic expansion of the Hankel function.

The asymptotic expansion of the Hankel function of the first kind for small argument is found from Ref. 12, Eqs. 9.1.3, 9.1.12 and 9.1.13:

$$H_0^{(1)}(z)=\frac{2i}{\pi}\ln\left(\frac{z}{2}\right)+\left(1+\frac{2i\gamma}{\pi}\right)-\frac{2i}{\pi}\left(\frac{z}{2}\right)^2\ln\left(\frac{z}{2}\right)-\left(1+\frac{2i(\gamma-1)}{\pi}\right)\left(\frac{z}{2}\right)^2+\ldots, \tag{51}$$

where γ is Euler's constant. From Eq. (51), a series solution for p of the following form is anticipated:

$$p=\sum_{n=0}^{\infty}[\mu_n(k)+\eta_n(k)], \tag{52}$$

where $\mu_n(k)=a_nk^n\ln k$, $\eta_n(k)=b_nk^n$ and a_n and b_n are constant coefficients. The terms in this series obey the following ordering criteria[13]:

$$\lim_{k\to 0}\frac{\eta_n(k)}{\mu_n(k)}=0,\ \lim_{k\to 0}\frac{\mu_{n+1}(k)}{\eta_n(k)}=0, \tag{53}$$

since $k^n/(k^n\ln\ k)\to 0$ and $k^{n+1}\ln\ k/k^n\to 0$ as $k\to 0$.

It is now possible to order Eq. (48). Write

$$J_x^{(i)}+\mathscr{L}p=0,\quad |z|<d, \tag{54}$$

where

$$\mathscr{L}\equiv\left\{\left[\frac{\partial^2}{\partial x^2}+k^2\right]\frac{i}{4}\int_{-d}^{d}dz'H_0^{(1)}(kr)\right\}_{x=x'=0}. \tag{55}$$

The way in which Eq. (55) is ordered depends on the form of Eq. (51). Substituting into Eq. (55) for $H_0^{(1)}$ from Eq. (51) and performing the double differentiation leads to the following definitions:

$$\mathscr{L}_{k^0}\equiv-\frac{1}{2\pi}\int_{-d}^{d}dz'\frac{1}{(z-z')^2}, \tag{56}$$

$$\mathscr{L}_{k^2\ln\ k}\equiv-\frac{k^2}{4\pi}\int_{-d}^{d}dz'\ln\left(\frac{k|z-z'|}{2}\right), \tag{57}$$

$$\mathscr{L}_{k^2}\equiv\left[1+\frac{i}{\pi}(2\gamma+1)\right]\frac{ik^2}{8}\int_{-d}^{d}dz'. \tag{58}$$

The terms $\mathscr{L}_{k^0\ln\ k}$, $\mathscr{L}_{k^1\ln\ k}$ and $\mathscr{L}_{k^1}$ are zero. For a uniform incident field directed normal to the crack face,

$$J_x^{(i)}=ikH_0e^{ik|z|}=H_0\left[ik+(ik)^2|z|+\frac{(ik)^3}{2!}|z|^2+\frac{(ik)^4}{3!}|z|^3+\ldots\right], \tag{59}$$

a simple power series expansion. It is found that, up to terms of order k^3 in the series solution for p, the ordering of Eq. (54) results in

$$J_{k^1}+\mathscr{L}_{k^0}p_{k^1}=0, \tag{60}$$

$$J_{k^2}+\mathscr{L}_{k^0}p_{k^2}=0, \tag{61}$$

$$(\mathscr{L}p)_{k^3\ln\ k}+\mathscr{L}_{k^0}p_{k^3\ln\ k}=0, \tag{62}$$

$$J_{k^3}+(\mathscr{L}p)_{k^3}+\mathscr{L}_{k^0}p_{k^3}=0. \tag{63}$$

The terms $(\mathscr{L}p)_{k^3\ln\ k}$ and $(\mathscr{L}p)_{k^3}$ are bracketed since contributions to them both arise from $\mathscr{L}_{k^2\ln\ k}p_{k^1}$. The ordering allows the Helmholtz problem of Eq. (54) to be dramatically simplified. Equations (60) to (63) essentially redefine the problem at each order in k so that the individual terms in the series expansion for p are each a solution of Laplace's equation. This simplification is fundamental to the perturbation method of solution. From Eqs. (60) to (63) it is clear that the first four non-zero terms in the series expansion for p are

$$p=p_{k^1}+p_{k^2}+p_{k^3\ln\ k}+p_{k^3}+\ \ldots\ . \tag{64}$$

We proceed to find these terms by linking Eqs. (60) to (63) with Eq. (23) at each order and using the dual integral equation method of solution given in Appendix A. From Eqs. (60) to (63) we identify

$$\mathscr{J}_{k^1}=\mu_0J_{k^1}, \tag{65}$$

$$\mathscr{J}_{k^2}=\mu_0J_{k^2}, \tag{66}$$

$$\mathscr{J}_{k^3\ln\ k}=\mu_0(\mathscr{L}p)_{k^3\ln\ k}, \tag{67}$$

$$\mathscr{J}_{k^3}=\mu_0[J_{k^3}+(\mathscr{L}p)_{k^3}], \tag{68}$$

and

$$\left(\frac{\partial^2\mathscr{A}}{\partial x^2}\right)_{k^i}=\mathscr{L}_{k^0}p_{k^i}. \tag{69}$$

The following solution, derived in Appendix A, is found:

$$\frac{\partial\mathscr{A}_x(x,z)}{\partial x}=\frac{2}{\pi}\int_0^{\infty}\int_0^{d}\int_0^{t}\frac{\mathscr{J}(w)}{\sqrt{t^2-w^2}}t\mathrm{J}_0(tu)e^{-ux}\times\cos(zu)dw\ \ dt\ \ du. \tag{70}$$

The terms in the series solution for p are found by evaluating Eq. (70) for each order using the appropriate ''current source'' given in Eqs. (65) to (68). The solution for p_{k^1} proceeds as follows. From Eqs. (65) and (59),

$$\mathscr{J}_{k^1}(z)=\mu_0H_0ik. \tag{71}$$

Substituting into Eq. (70) gives, for the inner integral,

$$\int_0^t\frac{1}{\sqrt{t^2-w^2}}dw=\frac{\pi}{2}. \tag{72}$$

Integration with respect to t in Eq. (70) is then performed by means of a simple change of variable and a standard result given in Ref. 12, Eq. 11.3.20, yielding

$$\left(\frac{\partial\mathscr{A}_x}{\partial x}\right)_{k^1}=\mu_0H_0ikd\int_0^{\infty}\frac{\mathrm{J}_1(ud)}{u}e^{-ux}\cos(zu)du. \tag{73}$$

Now write $\cos(zu)=\mathrm{Re}(e^{-izu})$ so that

$$\left(\frac{\partial\mathscr{A}_x}{\partial x}\right)_{k^1}=\mu_0H_0ikd\ \ \mathrm{Re}\left(\int_0^{\infty}\frac{\mathrm{J}_1(ud)}{u}e^{-\zeta u}du\right), \tag{74}$$

where $\zeta=x+iz$. Using another standard integral (Ref. 14, Eq. 6.623 no. 3) gives

$$\left(\frac{\partial\mathscr{A}_x}{\partial x}\right)_{k^1}=\mu_0H_0ikd\ \ \mathrm{Re}\left(\frac{\sqrt{\zeta^2+d^2}-\zeta}{d}\right), \tag{75}$$

from which it is simple, using Eq. (20), to find

$$p_{k^1}(z) = -2H_0 ik\sqrt{d^2-z^2}. \tag{76}$$

This highest order term in the series expansion of p behaves as expected; p_{k^1} is even in z and tends to zero at the crack edge.

The calculation of p_{k^2} proceeds similarly; by evaluating Eq. (70) with ''current source'' $\mathscr{J}_{k^2}$. From Eqs. (66) and (59),

$$\mathscr{J}_{k^2}(z) = -H_0 k^2 |z| \tag{77}$$

and the inner integral of Eq. (70) is now

$$\int_0^t \frac{w}{\sqrt{t^2-w^2}} dw = t. \tag{78}$$

The remaining two integrals in Eq. (70) are evaluated by noting that in the plane of the crack $e^{-ux}=1$ and reversing the order of integration. The resulting integral is (Ref. 14, Eq. 6.671 no. 2)

$$\int_0^\infty J_0(tu)\cos(zu)du = \frac{1}{\sqrt{t^2-z^2}}, \quad z<t, \tag{79}$$

which yields

$$\left[\left(\frac{\partial \mathscr{A}_x}{\partial x}\right)_{k^2}\right]_{x=0} = -\frac{2\mu_0 k^2 H_0}{\pi}\int_z^d \frac{t^2}{\sqrt{t^2-z^2}} dt, \quad z<t. \tag{80}$$

The integral of Eq. (80) may be evaluated by the change of variable $t=z \cosh x$. From Eq. (20) we finally obtain

$$p_{k^2}(z) = \frac{2H_0 k^2}{\pi} d\left[\sqrt{d^2-z^2} + \frac{z^2}{d}\ln\left(\frac{d+\sqrt{d^2-z^2}}{z}\right)\right]. \tag{81}$$

In order to find the terms $p_{k^3 \ln k}$ and p_{k^3}, the current sources $\mathscr{J}_{k^3 \ln k}$ and $\mathscr{J}_{k^3}$ are required. The details of their calculation are given in Appendix B. It is found that

$$\mathscr{J}_{k^3 \ln k} = \frac{\mu_0 H_0 ik^3 d^2}{4}\ln\left(\frac{kd}{4}\right), \tag{82}$$

$$\mathscr{J}_{k^3} = \frac{\mu_0 H_0 ik^3}{4}\left[d^2\left(1+\gamma-\frac{i\pi}{2}\right) - z^2\right]. \tag{83}$$

In calculating $p_{k^3 \ln k}$ the integrations follow as for p_{k^1} since $\mathscr{J}_{k^3 \ln k}$ has no z-dependence:

$$p_{k^3 \ln k} = -\frac{H_0 ik^3}{2}\ln\left(\frac{kd}{4}\right) d^2\sqrt{d^2-z^2}. \tag{84}$$

Lastly, p_{k^3} is found by evaluating Eq. (70) for the current source given by Eq. (83). The first term in $\mathscr{J}_{k^3}$ has no z-dependence and the integrations follow as for p_{k_1}. The second term, however, leads to integrals which have not been dealt with previously. The inner integral in Eq. (70) for the second term in Eq. (83) is easily evaluated:

$$\int_0^t \frac{w^2}{\sqrt{t^2-w^2}} dw = \frac{\pi t^2}{4}. \tag{85}$$

The integral with respect to t is then

$$\int_0^d t^3 J_0(tu)dt. \tag{86}$$

This may be solved by means of the standard integral given in Ref. 12, Eq. 11.1.1. The following is obtained:

$$\int_0^d t^3 J_0(tu)dt = \frac{d^3}{u}\frac{1}{\Gamma(-1)}\sum_{k=0}^{\infty}(2k+1) \times \frac{\Gamma(k-1)}{\Gamma(k+3)} J_{2k+1}(ud). \tag{87}$$

The reciprocal of the gamma function, $1/\Gamma(z)$, is an entire function which possesses simple zeros at the points for which z is an integer less than or equal to zero. This means that $1/\Gamma(-1)$ is zero and hence all terms in the series for which $k \geqslant 2$ vanish. Only the first two terms of the series need be considered, therefore, Equation (87) reduces to

$$\int_0^d t^3 J_0(tu)dt = \frac{d^3}{u}\frac{1}{\Gamma(-1)}\left[\frac{\Gamma(-1)}{\Gamma(3)} J_1(ud) + \frac{3\Gamma(0)}{\Gamma(4)} J_3(ud)\right]. \tag{88}$$

Now $\Gamma(3)=2$, $\Gamma(4)=6$ and

$$\lim_{\varepsilon\to 0}\frac{\left(\frac{1}{\Gamma(-1+\varepsilon)}\right)}{\left(\frac{1}{\Gamma(\varepsilon)}\right)} = -1,$$

which means that

$$\int_0^d t^3 J_0(tu)dt = \frac{d^3}{2u}[J_1(ud) - J_3(ud)]. \tag{89}$$

The only integral in Eq. (70) which remains to be evaluated is that with respect to u. This is performed by defining $\zeta = x+iz$ (as in the case of p_{k^1}), using the standard integral of Ref. 14, Eq. 6.623 no. 3, and taking the real part of the result. Using Eq. (20) gives

$$p_{k^3}(z) = -\frac{H_0 ik^3}{2}\sqrt{d^2-z^2}\left[d^2\left(\gamma+\frac{5}{6}-\frac{i\pi}{2}\right) - \frac{z^2}{3}\right]. \tag{90}$$

To summarise, the equivalent current dipole density on a two-dimensional surface-breaking crack has been found, in the low-frequency limit, as an asymptotic series whose first four non-vanishing terms are given in Eqs. (76), (81), (84) and (90).

B. Impedance

Equation (4) becomes, for this geometry,

$$\Delta Z = -\frac{1}{\sigma I^2}\int_{-d}^0 J_x^{(i)}(z)p(z)dz. \tag{91}$$

It can readily be deduced from Eqs. (59), (64) and (91) that the first four non-vanishing terms in the asymptotic series for ΔZ will be

$$\Delta Z = Z_{k^2} + Z_{k^3} + Z_{k^4 \ln k} + Z_{k^4} + \dots, \tag{92}$$

where

$$Z_{k^2} = -\frac{1}{\sigma I^2}\int_{-d}^{0} J_{k^1} p_{k^1} dz, \tag{93}$$

$$Z_{k^3} = -\frac{1}{\sigma I^2}\int_{-d}^{0} [J_{k^2} p_{k^1} + J_{k^1} p_{k^2}] dz, \tag{94}$$

$$Z_{k^4 \ln k} = -\frac{1}{\sigma I^2}\int_{-d}^{0} J_{k^1} p_{k^3 \ln k} dz, \tag{95}$$

$$Z_{k^4} = -\frac{1}{\sigma I^2}\int_{-d}^{0} [J_{k^3} p_{k^1} + J_{k^2} p_{k^2} + J_{k^1} p_{k^3}] dz. \tag{96}$$

The integrals obtained by substituting for J and p into Eqs. (93) to (96) are straightforward to evaluate. Those worthy of note are only those which involve p_{k^2}. In that case, evaluation may be performed by splitting the logarithm into a sum, integrating by parts and making the change of variable $z = d\sin t$. It is found that

$$\int_{-d}^{0} z^2 \ln\left(\frac{d+\sqrt{d^2-z^2}}{z}\right) dz = \frac{\pi d^3}{12}, \tag{97}$$

$$\int_{-d}^{0} z^3 \ln\left(\frac{d+\sqrt{d^2-z^2}}{z}\right) dz = \frac{-d^4}{6}. \tag{98}$$

The terms in the series expansion of ΔZ are found to be

$$Z_{k^2} = -\frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2 \frac{\pi}{2}(kd)^2, \tag{99}$$

$$Z_{k^3} = -\frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2 \frac{4i}{3}(kd)^3, \tag{100}$$

$$Z_{k^4 \ln k} = -\frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2 \frac{\pi}{8}(kd)^4 \ln\left(\frac{kd}{4}\right), \tag{101}$$

$$Z_{k^4} = \frac{1}{\sigma}\left(\frac{H_0}{I}\right)^2 \left[\frac{i\pi^2}{16} - \frac{\pi}{8}\left(\gamma + \frac{1}{4}\right) + \frac{1}{\pi}\right](kd)^4. \tag{102}$$

There is no reason why, in principle, this solution should not be extended and higher order terms evaluated. The results obtained so far, however, indicate that the series for p and ΔZ are rapidly converging and little improvement in accuracy would be obtained by proceeding to higher order.

VI. DISCUSSION OF RESULTS

The terms in the series expansion for the impedance change for the semi-circular, surface-breaking crack, given in Eqs. (44) and (45), agree with those obtained independently by Nair and Rose,[5] who also used a formulation valid for arbitrary frequency and then considered the low-frequency asymptotics. While our lowest order term in this series agrees with that of Kincaid,[3] the second term differs by a factor of 5/6. This discrepancy is due to the approximations made by Kincaid in deriving his result.

The analytical results for the dipole density and impedance change for the long, surface-breaking crack are compared with numerical predictions in the following figures. The numerical calculation is based on a vector-potential integral formulation. Details of the vector potential formulation and its numerical implementation are given elsewhere,[11] with its verification in the high-frequency regime. In Figs. 1 and 2 respectively, the real and imaginary parts of the dipole density are compared for a crack of depth 1.0 mm and for $d/\delta = 0.1$ and 0.2. Very good agreement between the analytical and numerical solutions is observed. In Fig. 3, the real and imaginary parts of the normalized impedance change, which is defined by $\sigma(I/H_0)^2\Delta Z$, are compared for d/δ in the range 0.0 to 0.4. For $d/\delta = 0.4$ there is approximately 20% difference between analytically and numerically calculated values of $\mathrm{Re}(\Delta Z)$. Agreement between the two sets of results is, however, extremely good for $\mathrm{Im}(\Delta Z)$, even beyond $d/\delta = 0.4$. In the numerical calculation, the crack was divided into 20 sections and a conductivity of $\sigma = 10^6$ S m^{-1} was used. Note that the lowest order terms in the impedance series expansions, Z_{k^2}, are purely imaginary for both examples considered here and the term of next order, Z_{k^3}, has real and imaginary parts of equal magnitude. This means that, if only two terms in the series are available, then

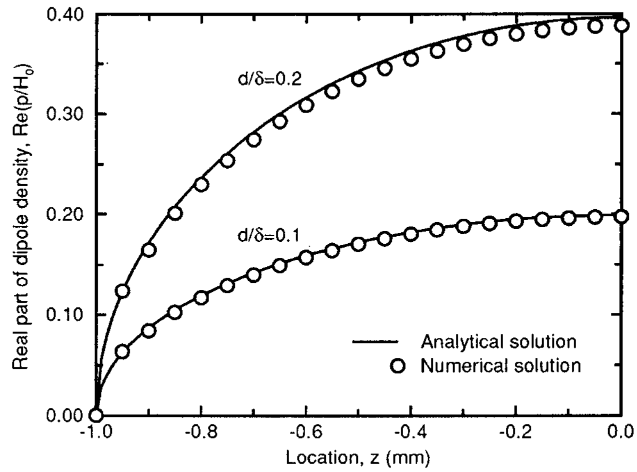

FIG. 1. Comparison between the real part of the analytical solution for the dipole density and the numerically calculated value for a long, surface-breaking crack in the low-frequency limit.

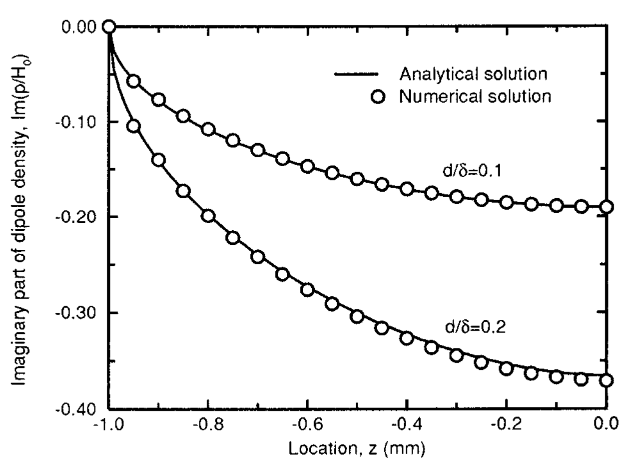

FIG. 2. Comparison between the imaginary part of the analytical solution for the dipole density and the numerically calculated value for a long, surface-breaking crack in the low-frequency limit.

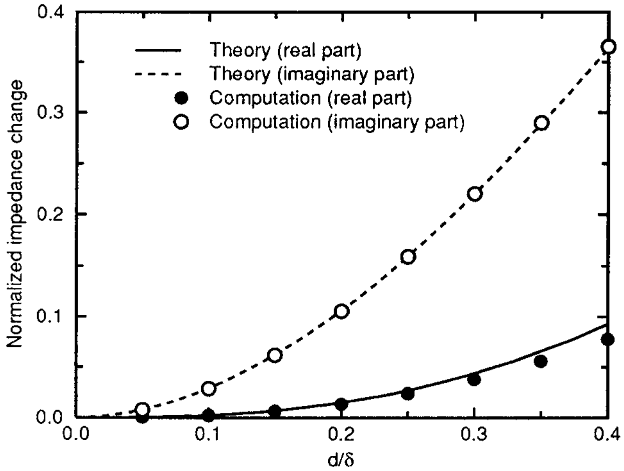

FIG. 3. Comparison between the analytical solution and the numerically calculated value for the impedance change due to a long, surface-breaking crack in the low-frequency limit.

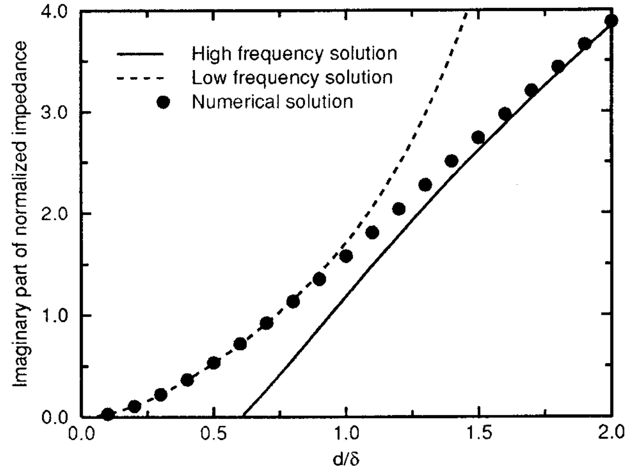

FIG. 5. Analytical and numerical solutions for the imaginary part of the impedance change due to a long, surface-breaking crack.

$\mathrm{Im}(\Delta Z)$ is known more accurately than $\mathrm{Re}(\Delta Z)$ since both terms contribute to $\mathrm{Im}(\Delta Z)$ but only the second term contributes to $\mathrm{Re}(\Delta Z)$. Even for a larger number of terms it follows that, since the dominant term in the series is purely imaginary, the analytical predictions for $\mathrm{Im}(\Delta Z)$ will be more accurate than those for $\mathrm{Re}(\Delta Z)$. This feature is observed in Fig. 3.

In Figs. 4 and 5 we include, for interest, comparisons of numerical and analytical predictions for a long, surface-breaking crack with d/δ up to 2.0 in value. (For $d/\delta > 2.0$ the analytical curve matches numerical predictions extremely well.) The analytical curve for low-frequency is that derived in this paper. The analytical predictions for higher frequencies are derived using a method based on the Geometrical Theory of Diffraction[15] and presented elsewhere.[16] In Fig. 4, the real part of ΔZ is considered. From the figure it is clear that the low-frequency theory works well for d/δ less than about 0.5 and the theory for higher frequencies works well for d/δ greater than about 1.0. This leaves only the small intermediate range of frequencies described by $0.5 < d/\delta < 1.0$ in which neither theory works particularly well. From Fig. 5, in which predictions for $\mathrm{Im}(\Delta Z)$ are shown, it is seen that the low-frequency theory works well for $d/\delta \leqslant 1.0$ (the numerical and analytical predictions agree to within 8% even for $d/\delta = 1.0$) and the theory for higher frequencies works well for $d/\delta \geqslant 1.5$. For the imaginary part of the impedance change, therefore, acceptable analytical solutions exist for all frequencies except those described by $1.0 < d/\delta < 1.5$.

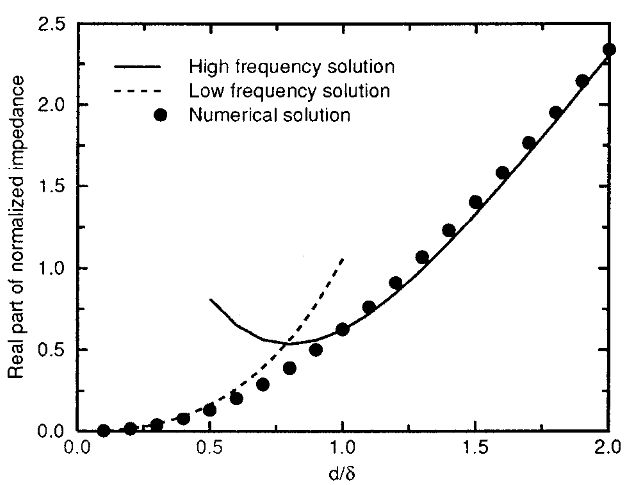

FIG. 4. Analytical and numerical solutions for the real part of the impedance change due to a long, surface-breaking crack.

VII. CONCLUSION

In this paper, we present a method by which low-frequency solutions for the impedance change due to closed cracks in metals can be calculated. The crack is treated as an equivalent layer of current dipoles whose field is represented by an integral over the dipole layer with the appropriate Green's function kernel. The formulation is valid for arbitrary frequency and the low-frequency solution is found using a perturbation method in which the individual terms in the series expansion for the dipole density distribution are each solutions of Laplace's equation. Two example solutions are given. The first two terms in the series expansion for ΔZ are calculated for a semi-circular, surface-breaking crack. The solution agrees with that of an independent calculation.[5] The first four terms in the series expansion for ΔZ are calculated for a long, uniformly deep, surface-breaking crack. The solution agrees well with that of an independent numerical calculation.[11]

Clearly, this method is suitable for predicting ΔZ for closed cracks of geometry other than those considered here. Both surface-breaking and sub-surface cracks can be treated. From Figs. 4 and 5, in which analytical solutions for the impedance change due to a long crack at both low (presented here) and higher frequencies (presented in Ref. 16), are compared with numerical predictions, we conclude that satisfactory analytical solutions now exist for most of the range of values of d/δ. The analytical solutions fail only in the small ranges given by $0.5<d/\delta<1.0$ for $\mathrm{Re}(\Delta Z)$ and $1.0<d/\delta<1.5$ for $\mathrm{Im}(\Delta Z)$.

ACKNOWLEDGMENTS

One of the authors (YY) would like to thank the British Council for financial support. JRB is supported by the Ministry of Defence, UK and the Defence Research Agency, Farnborough.

APPENDIX A: DUAL INTEGRAL EQUATION ANALYSIS FOR A LONG, SURFACE-BREAKING CRACK

The long crack problem can be formulated in terms of the following boundary conditions:

$$f(z)+\frac{\partial^2 \mathscr{A}_x(x,z)}{\partial x^2}=0, \quad x=0,|z|<d, \tag{A1}$$

$$\frac{\partial \mathscr{A}_x(x,z)}{\partial x}=0, \quad x=0,|z|>d, \tag{A2}$$

where $f(z)$ is the prescribed incident electric field and $\mathscr{A}_x(x,z)$ is the x-component of the magnetic vector potential of the ordered problem in which $\partial\mathscr{A}_x(x,z)/\partial x$ is a solution of Laplace's equation. Since $\partial\mathscr{A}_x/\partial x$ is an even function with respect to z, it is anticipated that the general integral representation will contain a cosine kernel:

$$\frac{\partial \mathscr{A}_x(x,z)}{\partial x}=\int_0^\infty \alpha(u)e^{-ux}\cos(zu)du. \tag{A3}$$

This expression is clearly a solution of Laplace's equation. Applying the boundary conditions to Eq. (A3) yields the following dual integral equations:

$$\int_0^\infty u\alpha(u)\cos(zu)du=f(z) \quad |z|<d, \tag{A4}$$

$$\int_0^\infty \alpha(u)\cos(zu)du=0 \quad |z|>d. \tag{A5}$$

These are equations of the Titchmarsh type, which have been considered in the following form in Ref. 6, Sec. 4.5:

$$\int_0^\infty u^{2k-1}\alpha(u)\cos(xu)du=F(x) \quad 0\leqslant x<1, \tag{A6}$$

$$\int_0^\infty \alpha(u)\cos(xu)du=0 \quad x>1. \tag{A7}$$

Only the cases in which $k=0$ or 1 are of physical interest; $k=1$ corresponds to the problem currently under consideration.

If we write

$$\Phi(z)=\int_0^\infty u\alpha(u)\cos(zu)du, \tag{A8}$$

$$\chi(z)=\int_0^\infty \alpha(u)\cos(zu)du, \tag{A9}$$

then the dual integral equations given in Eqs. (A4) and (A5) become

$$\Phi(z)=f(z) \quad |z|<d, \tag{A10}$$

$$\chi(z)=0 \quad |z|>d, \tag{A11}$$

for which an elementary solution can be found by a method similar to that for the problem of the electrified disc; Beltrami's method (Ref. 6, Sec. 3.5).

The function $\alpha(u)$ will be represented in terms of a function $g(t)$ by means of the relation

$$\alpha(u)=\int_0^d g(t)\mathrm{J}_0(tu)dt. \tag{A12}$$

Substituting Eq. (A12) into Eq. (A9) and reversing the order of integration yields

$$\chi(z)=\int_0^d g(t)\int_0^\infty \mathrm{J}_0(tu)\cos(zu)du \;\; dt. \tag{A13}$$

It is found that (Ref. 6, Eq. 2.1.13)

$$\chi(z)=\begin{cases}\displaystyle\int_z^d \frac{g(t)}{\sqrt{t^2-z^2}}dt & |z|<d\\ 0 & |z|>d.\end{cases} \tag{A14}$$

From Eq. (A8) we can write

$$\Phi(z)=\frac{\partial}{\partial z}\int_0^\infty \alpha(u)\sin(zu)du. \tag{A15}$$

Substituting for $\alpha(u)$ and interchanging the order of integration as before gives

$$\Phi(z)=\frac{\partial}{\partial z}\int_0^d g(t)\int_0^\infty \mathrm{J}_0(tu)\sin(zu)du \;\; dt. \tag{A16}$$

Integrating with respect to u (Ref. 6, Eq. 2.1.14) gives

$$\Phi(z)=\begin{cases}\displaystyle\frac{\partial}{\partial z}\int_0^z \frac{g(t)}{\sqrt{z^2-t^2}}dt & |z|<d\\ \displaystyle\frac{\partial}{\partial z}\int_0^d \frac{g(t)}{\sqrt{z^2-t^2}}dt & |z|>d.\end{cases} \tag{A17}$$

It follows that Eq. (A12) will give a solution of the dual integral equations provided that $g(t)$ is a solution of the integral equation

$$\frac{\partial}{\partial z}\int_0^z \frac{g(t)}{\sqrt{z^2-t^2}}dt=f(z) \quad |z|<d. \tag{A18}$$

The function $g(t)$ is given by (Ref. 6, Eq. 2.3.7)

$$g(t)=\frac{2t}{\pi}\int_0^t \frac{f(w)}{\sqrt{t^2-w^2}}dw. \tag{A19}$$

The solution of the dual integral Eqs. (A4) and (A5) is, therefore, given by Eq. (A12), with $g(t)$ given by Eq. (A19).

Substituting these results into the general integral representation for $\partial\mathscr{A}_x(x,z)/\partial x$, Eq. (A3), the following triple integral is obtained:

$$\frac{\partial \mathscr{A}_x(x,z)}{\partial x} = \frac{2}{\pi}\int_0^\infty \int_0^d \int_0^t \frac{f(w)}{\sqrt{t^2-w^2}} t\mathrm{J}_0(tu) e^{-ux} \times \cos(zu) dw \; dt \; du. \tag{A20}$$

This integral is evaluated for the prescribed incident field, $f(w)$, and, from the result, the equivalent current dipole density on the crack can be found.

APPENDIX B: DETERMINATION OF CURRENT SOURCES

In order to find the terms $p_{k^3 \ln k}$ and p_{k^3}, the ''current sources'' $\mathscr{J}_{k^3 \ln k}$ and $\mathscr{J}_{k^3}$ are required. Evaluation of $\mathscr{L}_{k^2 \ln k} p_{k_1}$ reveals terms of order both $k^3 \ln k$ and k^3. Using $\mathscr{L}_{k^2 \ln k}$ and p_{k^1}, given by Eqs. (57) and (76) respectively, it emerges that the determination of $\mathscr{L}_{k^2 \ln k} p_{k^1}$ involves evaluation of an integral of the kind:

$$\int_{-d}^{d} \sqrt{1-(z'/d)^2} \; \ln\sqrt{z^2-z'^2} dz'. \tag{B1}$$

Rewriting Eq. (B1) as follows,

$$\int_{-d}^{d} \frac{[1-(z'/d)^2]}{\sqrt{1-(z'/d)^2}} \ln\sqrt{z^2-z'^2} dz', \tag{B2}$$

and recognising that $[1-(z'/d)^2] = \frac{1}{2}[\mathrm{T}_0(z'/d) - \mathrm{T}_2(z'/d)]$ (where the T_n are Chebyshev polynomials of the first kind and $\mathrm{T}_0(x)=1$, $\mathrm{T}_2(x)=2x^2-1$) enables the evaluation of Eq. (B1) by means of the following relation (Ref. 17, Sec. 4.9).

$$-\frac{1}{\pi}\int_{-a}^{a} \frac{1}{\sqrt{1-(x'/a)^2}} \mathrm{T}_n\left(\frac{x'}{a}\right) \ln\sqrt{x^2-x'^2} dx' = \begin{cases} a\ln\left(\frac{2}{a}\right)\mathrm{T}_0\left(\frac{x}{a}\right) & n=0, \\ \left(\frac{a}{n}\right)\mathrm{T}_n\left(\frac{x}{a}\right) & n=1,2,\ldots\ldots \end{cases} \tag{B3}$$

This gives

$$\mathscr{L}_{k^2 \ln k} p_{k^1} = \frac{H_0 i k^3 d^2}{4}\left[\ln\left(\frac{kd}{4}\right) + \frac{1}{2} + \left(\frac{z}{d}\right)^2\right]. \tag{B4}$$

The term of order $k^3 \ln k$ is clearly

$$(\mathscr{L}p)_{k^3 \ln k} = \frac{H_0 i k^3 d^2}{4} \ln\left(\frac{kd}{4}\right), \tag{B5}$$

and, from Eq. (67),

$$\mathscr{J}_{k^3 \ln k} = \frac{\mu_0 H_0 i k^3 d^2}{4} \ln\left(\frac{kd}{4}\right). \tag{B6}$$

The remaining terms in Eq. (B4) contribute to $(\mathscr{L}p)_{k^3}$. The evaluation of $\mathscr{L}_{k^2} p_{k^1}$ from Eqs. (58) and (76) is straightforward and, combined with the appropriate terms from Eq. (B4) gives

$$(\mathscr{L}p)_{k^3} = \frac{H_0 i k^3}{4}\left[d^2\left(1+\gamma-\frac{i\pi}{2}\right) + z^2\right]. \tag{B7}$$

From Eq. (68) with Eqs. (59) and (B7) we find

$$\mathscr{J}_{k^3} = \frac{\mu_0 H_0 i k^3}{4}\left[d^2\left(1+\gamma-\frac{i\pi}{2}\right) - z^2\right]. \tag{B8}$$

[1] M. L. Burrows, Ph.D. dissertation, The University of Michigan, University Microfilms International, Ann Arbor, Michigan, 1964.
[2] T. G. Kincaid, K. Fong, and M. V. K. Chari, Proceedings of the DARPA/AFML Review of Progress in Quantitative NDE, Rockwell International Report to Air Force Materials Laboratory, 1979, pp. 463–468.
[3] T. G. Kincaid, in *Review of Progress in Quantitative NDE*, edited by D. O. Thompson and D. E. Chimenti (Plenum, New York, 1982), Vol. 1, pp. 355–356.
[4] B. A. Auld, F. G. Muennemann, and M. Riaziat, in *Research Techniques in Nondestructive Testing*, edited by R. S. Sharpe (Academic, London, 1984), Vol. 7, Chap. 2.
[5] S. M. Nair and J. H. Rose, J. Appl. Phys. **70**, 1924 (1991)
[6] I. N. Sneddon, *Mixed Boundary Value Problems in Potential Theory* (North Holland, Amsterdam, 1966).
[7] R. F. Harrington, *Time Harmonic Electromagnetic Fields* (McGraw–Hill, New York, 1961).
[8] J. R. Bowler, J. Appl. Phys. **75**, 8128 (1994)
[9] C.-T. Tai, *Dyadic Green Functions in Electromagnetic Theory*, 2nd ed. (IEEE, New York, 1993).
[10] A. P. Raiche and J. H. Coggon, Geophys. J. R. Astron. Soc. **42**, 1035 (1975).
[11] J. R. Bowler, Y. Yoshida and N. Harfield, IEEE Trans. Mag. (submitted).
[12] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[13] J. D. Murray, *Asymptotic Analysis* (Clarendon, Oxford, 1974).
[14] *Table of Integrals, Series and Products*, edited by I. S. Gradshteyn and I. M. Ryzhik, 4th ed. (Academic, New York and London, 1965).
[15] J. B. Keller, J. Opt. Soc. Am. **52**, 116 (1962).
[16] N. Harfield and J. R. Bowler, Proc. R. Soc. London Ser. A (in print).
[17] J. Van Bladel, *Singular Electromagnetic Fields and Sources* (Clarendon, Oxford, 1991).