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Evaluation of the magnetic field near a crack with application to magnetic particle inspection

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Abstract

In this paper, the magnetic field at the mouth of a crack in ferromagnetic steel is determined by means of a two-dimensional, linear model. The solution is found by employing an analytical method in which one complex variable is transformed into another by means of a mapping function. An approximate boundary condition, based on the fact that the steel permeability is much larger than that of free space, is used. In this way, three representations of a crack are treated: narrow and open cracks and a semi-elliptical indentation. The mapping function transforms these shapes into a half-plane geometry for which the solution is easily obtained. The advantage of this analytical approach is that the results are readily accessible without the need for a large numerical code. Example calculations are compared with each other and with calculations based on a former theory. This work has application in electromagnetic non-destructive evaluations: eddy-current testing, flux leakage measurements and, most directly, magnetic particle inspection.

1. Introduction

This paper describes a theoretical and computational model which has been developed to study the interaction of magnetic fields with cracks in ferromagnetic steels. The theory finds application in a number of areas in non-destructive evaluation, including magnetic flux leakage measurements and eddy-current testing, but its primary application is to magnetic particle inspection (MPI). MPI is widely used as a very sensitive method of detecting surface-breaking cracks in ferritic steel. In MPI, ink containing magnetic particles in the form of a colloidal suspension is applied to the surface of a magnetized test-piece. The magnetic particles diffuse through the liquid under the influence of an externally applied magnetic field until a steady state is reached. They accumulate in regions where the magnetic field is greatest, such as at the mouth of a crack. In a contribution to the theory of MPI, the magnetic field is here evaluated for a number of two-dimensional configurations that are of practical relevance.

Many factors influence the magnetic field produced in a typical inspection using magnetic particles. These include the geometry of the magnet yoke, the magnet coil parameters and the shape of the component under test. However, essential knowledge of the magnetic field in the vicinity of a crack can be gained without reference to the details of the magnetic circuit. Instead, it is useful to focus on a region near to the flaw and obtain a local field solution which may then be scaled if necessary with reference to the larger magnetic circuit.

In this paper, a field perturbation due to a localized flaw subjected to a uniform unperturbed field is considered. A class of such problems is defined by considering the field in a homogeneous, permeable half-space perturbed by a surface irregularity. Members of the class are distinguished by the nature of the flaw, which may be an ideal, thin crack, a semi-elliptical indentation, etc. Here it is assumed that the field and flaw are invariant in one dimension. The problem domain then reduces to two dimensions and the region of the metal is represented by a half-plane with an irregularity on its line boundary.

The magnetic field in the vicinity of the defect is calculated in two stages. In the first stage, the magnetic field in the steel is determined. This solution defines the magnetic potential on the air–conductor interface from which the magnetic field in air above the metal is then evaluated. The problem is formulated in terms of complex potentials and the method of conformal

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transformations is used to transform the flawed domain into a simpler, un-flawed domain in which the solution is easily obtained. This approach yields analytic results which are readily accessible without the need for a large numerical code.

Three solutions are presented corresponding to three representations of a crack: narrow and open cracks and a semi-elliptical indentation. In the cases of the narrow crack and the semi-elliptical indentation, the field perturbation due to the crack is found in response to an applied magnetic field with magnitude $1\,\mathrm{A\,m^{-1}}$. In the case of the open crack, the solution is truly local; only the field perturbation in the region of the crack mouth is considered. In order to obtain a solution, it is assumed that the magnetic potential difference across the crack mouth has unit magnitude. This assumption is not restrictive since simple scaling of the result will give the correct magnitude for other applied field strengths. Calculations of the field at the crack mouth are compared with each other and with results of a former theory due to Edwards and Palmer [1], in turn, based on the earlier work of Zatsepin and Shcherbinin [2].

2. Formulation

Under static conditions, the magnetic field in a current-free region can be written in terms of a magnetic scalar potential $\phi(x, y)$ defined such that

$$\boldsymbol{H} = -\nabla\phi \qquad \text{with} \qquad \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}, \tag{1}$$

where $\hat{x}$ and $\hat{y}$ are unit vectors. In linear homogeneous regions, the magnetic field, like the magnetic flux density, has zero divergence. Equating the divergence of (1) to zero shows that the potential satisfies the Laplace equation:

$$\nabla^2\phi = 0. \tag{2}$$

The problem of determining the magnetic field reduces to one of finding ϕ subject to suitable boundary conditions.

Solutions of the Laplace equation in two dimensions are commonly found using the properties of analytical functions of a complex variable [3]. A function $F = \phi + \mathrm{i}\psi$ of the complex variable $z = x + \mathrm{i}y$ is analytic in a certain domain if it is single-valued and differentiable at all points in the domain. In order to be differentiable, the real and imaginary parts of F are connected by the Cauchy–Riemann equations:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad \text{and} \qquad \frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y}.$$

The real function ψ, here called the stream function, is introduced to complement the potential ϕ. Then, using the Cauchy–Riemann equations,

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}z} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial z}\\ &= \frac{1}{2}\left(\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)\\ &= \frac{\partial\phi}{\partial x} - \mathrm{i}\frac{\partial\phi}{\partial y}\\ &= -H_x + \mathrm{i}H_y.\end{aligned} \tag{3}$$

In (3), the total derivative of the complex potential gives the components of the magnetic field.

In general, a transformation from the domain of a complex variable t to the domain of z is written as

$$z = M(t), \tag{4}$$

where M is a single-valued mapping function. Starting with, for example, a uniform field of unit magnitude, its representation in the t-domain is expressed in terms of a complex potential given simply by

$$f(t) = t. \tag{5}$$

Then the potential in the z-domain, which corresponds to ordinary configuration space, is

$$F(z) = M^{-1}(z). \tag{6}$$

A flawed half-plane problem can be solved by using a mapping to transform an 'unflawed domain' into a 'flawed domain'. The advantage of this approach is that the solution in the unflawed domain is typically more easy to obtain. It is sometimes the case that an explicit form for the inverse mapping function, which transforms the solution back into the flawed domain, is not available. If this is so, then to find t for a given z and evaluate the complex potential $F(z)$, equation (5), an alternative means of finding the inverse relationship is required. In this study, in the case of the open crack, t is found numerically ([4], see Brent's routine) for a given z by varying t until $|z - M(t)|$ is minimized.

3. Boundary conditions

The continuity of the normal component of the magnetic flux at the interface between two regions S_1 and S_2, is expressed as

$$\mu_1\left(\frac{\partial\phi}{\partial n}\right)_1 = \mu_2\left(\frac{\partial\phi}{\partial n}\right)_2. \tag{7}$$

Assuming the metal (region S_1, say) has a high relative permeability ($\mu_1 \gg \mu_2$), one can make an approximation for the boundary condition and proceed to solve a problem defined in the internal domain:

$$\nabla^2\phi = 0, \quad z \in S_1 \qquad \text{with } \frac{\partial\phi}{\partial n} = 0, \quad z \in \partial S_1, \tag{8}$$

where ∂S_1 denotes the boundary of region S_1. This approximation is good for many common ferromagnetic steels provided they are not saturated. The approximate interface condition derived from the continuity of the normal flux density allows the internal field to be decoupled from the external field. In this way, the former can be calculated independently of the latter. An underlying assumption of condition (8) is that the magnetic flux leakage has a negligible effect on the internal field in the metal. This means that the solutions obtained below are valid for cracks whose depth is not excessively greater than the width since, in practice, flux leaks between the faces of very deep, narrow cracks. Another consequence of equation (8) is that the permeability of the steel does not appear explicitly in the result. This approximation is a good one for most steels in which $\mu_1 \approx 100\mu_2$.

The continuity of the tangential magnetic field at the interface implies that the potential is continuous there:

$$(\phi)_1 = (\phi)_2. \tag{9}$$

Thus the internal field solution defines the potential on the external boundary. Let the boundary potential be g. Then the external field can be determined from the solution of the Laplace equation with a Dirichlet boundary condition:

$$\nabla^2\phi = 0, \quad z \in S_2 \qquad \text{with } \phi = g, \quad z \in \partial S_2. \tag{10}$$

Below, three solutions are considered in order of increasing complexity. In the first and second cases (narrow crack and semi-elliptical indentation), it is assumed that, if no crack is present, a uniform background magnetic field exists. With a crack present, an additional field arises which depends on the far field. This far magnetic field is normalized to unit magnitude ($1\ \mathrm{A\,m^{-1}}$) as expressed by the condition

$$\lim_{|z|\to\infty} \phi = -z. \tag{11}$$

In the third case (open crack), a local solution is sought corresponding to unit potential difference at the crack mouth.

4. Solutions for three flaw models

4.1. Narrow crack

In the first and simplest crack representation, the internal solution is expressed in terms of a complex potential $F(z) = -\sqrt{z^2+b^2}$ where b is the depth of the crack. The function gives rise to a uniform far field of $1\ \mathrm{A\,m^{-1}}$ and a magnetic potential difference across the crack mouth of magnitude $2b$. Consider a region in the vicinity of the mouth of a narrow crack with the internal field represented by the above complex potential. On a scale which is large compared with the crack opening but small compared with the crack depth, the crack opening can be neglected and the magnetic potential in the metal on both sides of the crack can be assumed locally constant. The difference between the constant values on each side of the crack will be referred to as the mouth potential difference (MPD). With an MPD of $2b$, the external magnetic scalar potential is given by

$$\phi = -\frac{2b\theta}{\pi} = -\frac{2b}{\pi}\arctan\left(\frac{x}{y}\right), \tag{12}$$

where θ is the angle between a line radiating from the line of the crack mouth and the y-axis. Taking the derivative of ϕ with respect to x and y, the magnetic field components are found to be

$$H_x = \frac{2b}{\pi}\frac{y}{x^2+y^2} \qquad \text{and} \qquad H_y = -\frac{2b}{\pi}\frac{x}{x^2+y^2}. \tag{13}$$

These equations give a reasonable estimate of the field in air at a distance from the crack mouth that is large compared with its opening but small compared with its depth.

4.2. Semi-elliptical crack

The transformation [5]

$$z = a(t + \lambda\sqrt{t^2-1}), \tag{14}$$

where a and λ are constants, maps the upper half of the t-plane into the upper half of the z-plane indented by a semi-elliptical region (figure 1). The parameter a is the length of the semi-minor axis of the ellipse and λ is the ratio of the semi-major axis to the semi-minor axis. The inverse transform is given by

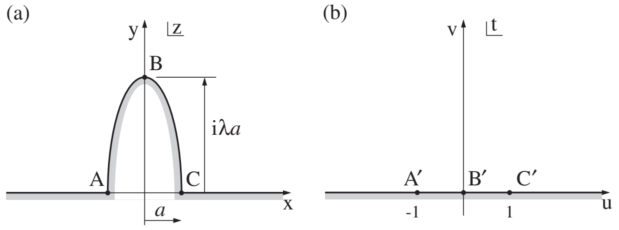

Figure 1. (*a*) Half-plane indented by a semi-ellipse mapped from (*b*) a half-plane.

$$t = \frac{a}{\beta^2}[\lambda\sqrt{z^2+\beta^2} - z], \qquad \text{where} \qquad \beta^2 = a^2(\lambda^2-1). \tag{15}$$

A uniform magnetic field is represented in the t-plane by the elementary function

$$f(t) = -a(\lambda+1)t, \tag{16}$$

where the coefficient $a(\lambda+1)$ ensures that the corresponding far field in the z-plane has unit magnitude. Transforming this solution to the z-plane using equation (15) gives

$$F(z) = \frac{z - \lambda\sqrt{z^2+\beta^2}}{\lambda-1}, \qquad z \in S_1 \tag{17}$$

for the internal complex potential in the z-plane. Taking the derivative of F with respect to z as shown in equation (3) gives the following equation for the magnetic field in the metal:

$$-H_x + \mathrm{i}H_y = \frac{1}{\lambda-1}\left[1 - \frac{\lambda z}{\sqrt{z^2+\beta^2}}\right], \qquad z \in S_1. \tag{18}$$

The external field is approximated by assuming that the field in the semi-elliptical region of the crack is uniform and x-directed. This assumption, together with the expression for the internal potential, equation (17), and the continuity condition on the scalar potential, equation (9), determines the complex potential at the line $x = 0$ and therefore in the upper half-plane. The complex potential in this region is written as [6]

$$F(z) = -z - \frac{1}{\mathrm{i}\pi}\int_{-\infty}^{\infty}\frac{h(s)}{z-s}\,\mathrm{d}s, \qquad z \in S_2, \tag{19}$$

where $h(x)$ is the perturbed potential at the line $x = 0$ due to the crack and is given by

$$h(x) = \begin{cases} \lambda x, & |x| \leqslant a, \\ \dfrac{\lambda}{\lambda-1}[\sqrt{x^2+\beta^2} - x], & |x| \geqslant a. \end{cases} \tag{20}$$

Note that $h(x)$ for $|x| \geqslant a$ is found from (17) by subtracting z and taking the real part with $y = 0$. From (19) and (20) it is found that the external complex potential is given by

$$\begin{aligned} F(z) = &-z - \frac{\lambda}{\mathrm{i}\pi}\left[z\log\left(\frac{z+a}{z-a}\right) - 2a\right] \\ &- \frac{\lambda}{\mathrm{i}\pi(\lambda-1)}\left[(z - \sqrt{z^2+\beta^2})\log\left(\frac{a+z}{a-z}\right)\right. \\ &\left. + \sqrt{z^2+\beta^2}\log\left(\frac{a\lambda\sqrt{z^2+\beta^2} + az - \beta^2}{a\lambda\sqrt{z^2+\beta^2} + az + \beta^2}\right)\right], \\ &z \in S_2. \end{aligned} \tag{21}$$

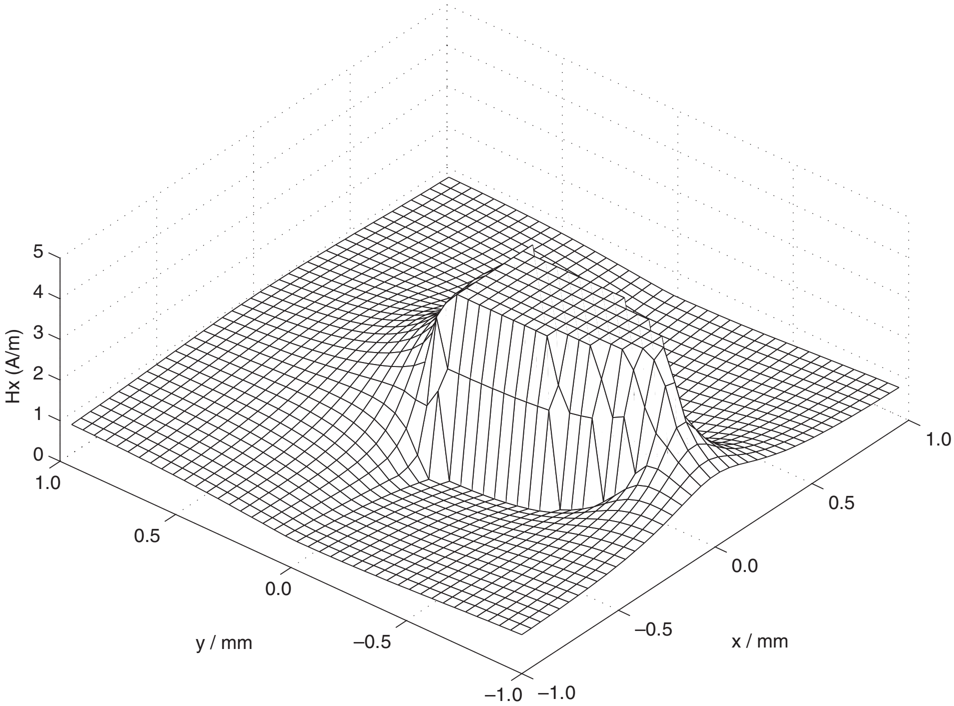

Figure 2. H_x in the presence of a long, semi-elliptical defect 0.5 mm wide and 0.75 mm deep, in a metal half-space. The crack mouth is centred at the origin and the metal occupies the region $y \leqslant 0$.

Taking the derivative of (21) gives

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}z} = & -1 - \frac{\lambda}{\mathrm{i}\pi}\left[\log\left(\frac{z+a}{z-a}\right) - \frac{2az}{z^2-a^2}\right] - \frac{\lambda}{\mathrm{i}\pi(\lambda-1)} \\ & \times\left[\frac{z}{\sqrt{z^2+\beta^2}}\log\left(\frac{a\lambda\sqrt{z^2+\beta^2}+az-\beta^2}{a\lambda\sqrt{z^2+\beta^2}+az+\beta^2}\right)\right. \\ & + \frac{a\beta^2(\lambda z+\sqrt{z^2+\beta^2})}{(a\lambda\sqrt{z^2+\beta^2}+az-\beta^2)(a\lambda\sqrt{z^2+\beta^2}+az+\beta^2)} \\ & + \left(1-\frac{z}{\sqrt{z^2+\beta^2}}\right)\log\left(\frac{a+z}{a-z}\right) \\ & \left. - \frac{2az(z-\sqrt{z^2+\beta^2})}{z^2-a^2}\right], \qquad z \in S_2, \end{aligned} \tag{22}$$

from which the components of the field are found by taking real and imaginary parts according to equation (3). This is easily accomplished computationally, but analytic expressions for H_x and H_y at the crack mouth can also be easily obtained by putting $z = 0$ in equation (22) and noting that $\log(z) = \log|z| + \mathrm{i}\alpha$ with $-\pi < \alpha \equiv \arg(z) \leqslant \pi$. It is found that, at the crack mouth,

$$H_x(0,0) = 1 + \lambda. \tag{23}$$

This result agrees with that of Edwards and Palmer [1]. The x-component of the magnetic field in the presence of a semi-elliptical defect, calculated from equations (18) and (22) for the metal and air regions, respectively, is shown in figure 2 for a defect with width 0.5 mm and depth 0.75 mm.

4.3. Open crack

In contrast to the two previous crack representations, this third representation accurately models the singularity in the magnetic field at the vertices of the crack mouth.

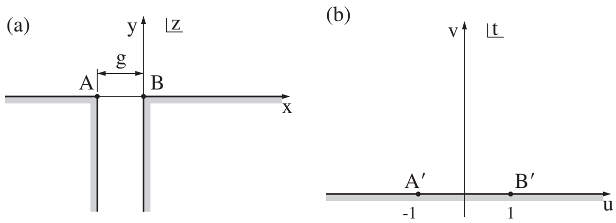

Figure 3. Mapping to (*a*) a slot from (*b*) a half-plane.

The open crack can be treated in two different ways. It may be treated either as a three-vertex system (figure 3) or as a two-vertex system [7] making use of symmetry. Applying the Schwarz–Christoffel theory [5] to the structure shown in figure 3 gives

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{t}(t^2-1)^{1/2}, \tag{24}$$

where $t = u + \mathrm{i}v$, $t = \pm 1$ corresponds to the vertices on either side of the crack mouth, and the point $t = 0$ maps to the bottom of the crack. By integration of (24), it is found that

$$z = \frac{g}{\pi}\left\{\sqrt{t^2-1} + \mathrm{i}\log\left[\frac{1+\mathrm{i}\sqrt{t^2-1}}{t}\right]\right\}. \tag{25}$$

In order to obtain the desired solution in the z-plane, consider a complex potential defined in the t-plane given by

$$f(t) = \frac{1}{\pi}\log(\mathrm{i}t) \tag{26}$$

and having the property

$$\mathrm{Re}\{f(u)\} = \begin{cases} \frac{1}{2}, & u < 0, \\ -\frac{1}{2}, & u > 0. \end{cases} \tag{27}$$

Mapping this potential to the z-plane produces a solution that is constant on the boundaries but maintains unit potential difference across the crack mouth. Because equation (25) cannot be inverted, the inverse mapping is carried out numerically as described in section 2.

4.4. Example calculations

In figure 4, a comparison is made between the magnetic field predicted by the three solutions presented here. The field values are computed for a set of points 0.1 mm above the surface of a sample containing a slot 0.2 mm wide and 1.0 mm deep. In the calculations, the result for the open crack is scaled to match the others in terms of potential difference at the crack mouth. Note that, because the size of the crack opening is not small compared with the stand-off distance of 0.1 mm, the elementary result given in equation (13) is not strictly applicable in this case. Nonetheless, there is reasonable agreement between the three sets of predictions at this height.

Closer to the surface of the metal, the open crack theory is the most accurate since it accounts for the singularity in the magnetic field at the metal vertices. This is demonstrated in figure 5 in which predictions are compared at a set of points 0.005 mm above the surface of a sample. The crack dimensions are the same as for figure 4. Comparisons are made between predictions of the semi-elliptical crack model, the open crack solution and the theory of Edwards and Palmer [1] in which the field is represented in terms of a uniform layer of magnetic monopoles at the crack faces. In calculating a result from the theory of [1], the high permeability limit is assumed.

5. Summary and conclusion

A number of solutions of the Laplace equation in two dimensions have been derived for evaluating the interaction of an imposed magnetic field with cracks in ferromagnetic steels. The solutions are obtained by conformal mapping which exploits the properties of analytical functions of a complex variable. The calculations predict the magnetic field at the mouth of the crack, allowing estimates of the forces experienced by magnetic particles in MPI.

A comparison is made between solutions found by conformal mapping and one found from a theory developed by Edwards and Palmer [1]. The elementary theory for a narrow crack is useful for calculating the magnetic field at a height above the sample which is large compared with the crack opening but small compared with its depth. The solution for a semi-elliptical indentation is not restricted in this way but makes the approximation that the magnetic field in the crack opening is uniform and x-directed. The theory for the open crack is the most accurate since it correctly represents the singularity in the field at the metal vertices. The theory of Edwards and Palmer [1] is based on the work by Zatsepin and Shcherbinin [2] who represent the magnetic field in terms of uniform magnetic monopole layers at the crack faces. The uniform monopole layers give a reasonable approximation of

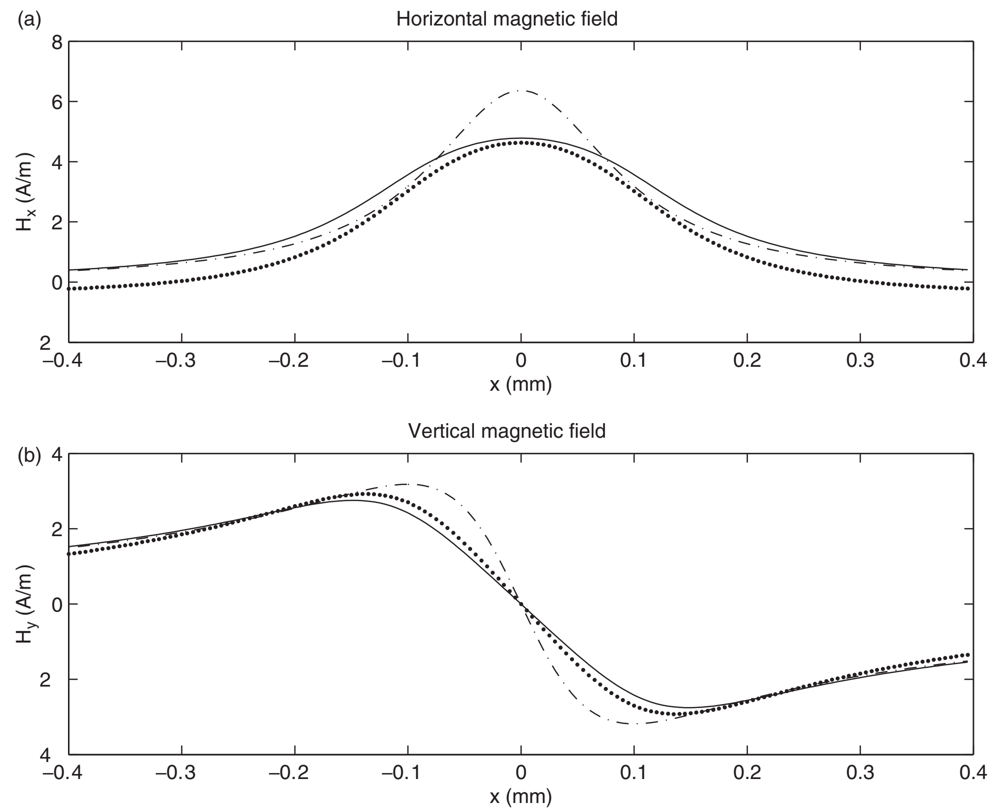

Figure 4. Magnetic field variation with x at $y = 0.1$ mm for a crack 0.2 mm wide and 1.0 mm deep: (— · —) narrow crack, (· · · · · ·) semi-elliptical indentation and (——) open crack.

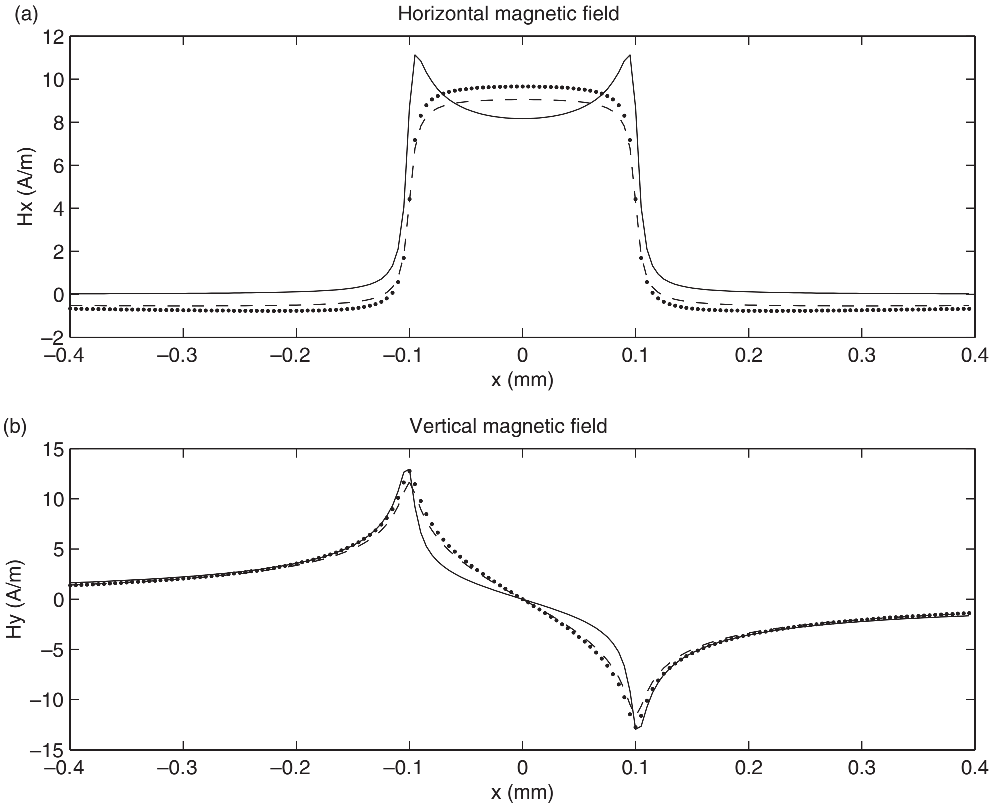

Figure 5. As for figure 4 but with $y = 0.005$ mm: (· · · · · ·) semi-elliptical indentation, (——) open crack and (- - - -) Edwards and Palmer [1].

the field but do not describe the corner singularity correctly, as is clear from figure 5.

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