Sampling Theory

With some ideas from

An Introduction to Mixed-Signal IC test and measurement

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About Chapter 6
Pages 147-185

- The Chapter introduces the reader to
  - Sampling theory
  - Coherent sampling
    - And how they apply to mixed signal testing
  - The level is designed for beginning of Mixed-signal test engineers so it is not in-depth, but is practical
  - Assumes you know how coherent multitone sample sets are created sourced and captured
  - The next Chapter will apply digital signal processing algorithms (FFT, power spectrum analysis, etc) allows possibility of low test time and high accuracy
- Why should we consider DSP methods: the speed and accuracy that is offered with this method is not possible with conventional methods

Sampling Theory

- Analog Measurements using DSP
  - Gain, frequency response etc. can be measured with purely analog instruments.
    - For instance gain, an AC continuous sine wave generator can be programmed to source a single tone at a desired voltage level, Vin, and a desired frequency. A true RMS voltmeter can then measure the output response from the DUT, Vout. The gain is then Gain=Vout/Vin.
  - Problems:
    - Relatively slow for multiple frequencies
    - Unable to measure distortion without removal of the fundamental tone with a notch filter - adding to the complexity of the DIB hardware.
    - Analog testing measures RMS noise, making results unrepeatable without an averaging or bandpass filter

Sampling Theory

- Traditional vs. DSP-Based Testing of AC Parameters
  - Digital Signal Processing (DSP)
  - Adopted by ATE industry in the early ‘80s.
  - Allows faster, more accurate and more repeatable measurements than traditional AC measurements
  - Strong background in signal processing theory is a definite need for the test engineer

Sampling and Reconstruction

- Continuous Time and Discrete Time
  - Analog signals are continuous time signals
    - Can exist as voltage or current waveforms, or as mathematical functions $v(t)=\sin(2\pi Ft)$
  - Modern systems must convert continuous time signals into discrete digital signals to be compatible with digital storage, digital transmission, and digital signal processing.
  - Sampling is the process in which continuous time signals are converted into discrete digital samples.
    - Sampling is usually performed at a constant frequency or sampling rate.

Sampling and Reconstruction

- **Continuous Time and Discrete Time - cont.**
  - Sampled waveforms can be defined as a sequence of numbers
  - Samples are uniformly spaced at intervals of a period T_s seconds
  - T_s is the sampling period and $F_s=1/\ T_s$ is the sampling frequency
  - Sampling instants are at $t = n\ T_s$, where n is integer multiples of T_s
  - In many practical manipulation the period T_s is assumed to be constant and equal to 1 without loss of generality.

Sampling and Reconstruction

- **Continuous Time and Discrete Time - cont.**
  - ADC's perform the sampling in real world circuits (they would bring their own limitations, short coming, and problems).
  - In the mathematical world, sampling is described by discrete equations like

    $v(n) = A\sin(2\pi Mn/N + \phi) = A\sin(2\pi n f_0/F_s + \phi)$

    where M, n and N are integers
  - These samples can be easily stored in memory and processed using DSP

Sampling and Reconstruction

- **Continuous Time and Discrete Time - cont.**
  - A sampled waveform can also be defined as a continuous function of time
  - This definition makes use of the delta functions
  - delta function:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad\qquad v_a(t) = \sum_{n=-\infty}^{\infty} v(t)\,\delta_{t,nT_s}(t - nT_S)$$

  - values of $v_a(t)$ are embedded in the area, not the amplitude

What about the Delta Function?

The Dirac delta, introduced by Paul Dirac(1902-1984) , can be thought of as a function $\delta(x)$ that has the value of infinity for $x = 0$, the value zero elsewhere, and a total integral of one. The graph of the delta function can be thought of as following the whole x-axis and the positive y-axis.

The Dirac delta is very useful as an approximation for tall narrow spike functions. It is the same type of abstraction as a point charge, point mass or electron point. For example, in calculating the dynamics of a baseball being hit by a bat, approximating the force of the bat hitting the baseball by a delta function is a helpful trick. In doing so, one not only simplifies the equations, but one also is able to calculate the motion of the baseball by only considering the total impulse of the bat against the ball rather than requiring knowledge of the details of how the bat transferred energy to the ball.

Properties of Dirac Delta

valid for any continuous function f.

As a distribution addressing a physical model, the Dirac delta was proposed as:

A distribution that expresses the derivative of the Heaviside step function

However, there is no function $\delta(x)$ with this property. Technically speaking, the Dirac delta is not a function but a distribution which is a mathematical expression that is well defined only when integrated.

What is a function

• The concept of function is a generalization of the common notion of a "mathematical formula". Functions describe special mathematical relationships between two objects, x and y=f(x). The object x is called the argument of the function f, and y is said to "depend functionally" on x.

•It should be noted that the words "function", "mapping", "map" and "transformation" are usually used synonymously. Furthermore, functions may occasionally referred to as well-defined function or total function (See the section "#Formal Definition" below).

•Intuitively, a function is a way to assign to each value of the argument x a unique value of the function f(x). This could be specified by a formula, a relationship, and/or a rule. This concept is deterministic, always producing the same result from the same input (the generalization to random values is called a stochastic process). A function may be thought of as a "machine" or "black box" converting valid input into a unique output.

http://www.encyclopedia4u.com/f/function.html

How many delta functions do we have?

There are two major and important delta functions

1. The Dirac: which is defined by the distribution that expresses the derivative of a step function $\delta(x)$

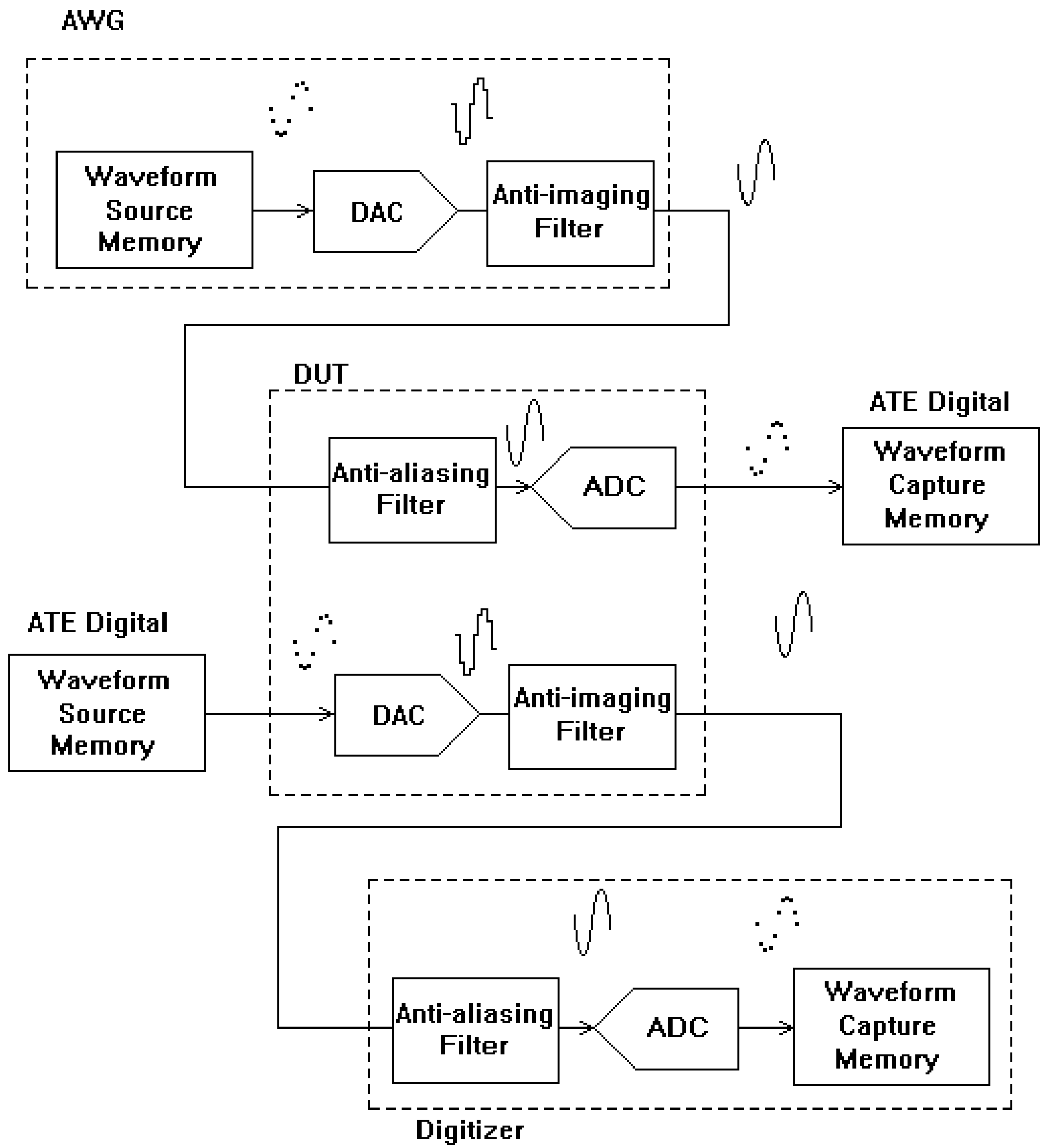
2. The Kronecker delta function δ_{nm} which is 1 if n=m and is 0 otherwise

Sampling and Reconstruction

- **Continuous Time and Discrete Time - cont.**
  - Reconstruction
    - The reverse process of sampling
    - Process by which a sampled waveform is converted into a continuous waveform
    - Reconstruction is the operation that fills in the missing waveform between samples
    - DAC's and an anti-imaging filter perform this function in mixed-signal applications They bring their limitations
    - The Discrete Fourier Transforms perform this function in math

Sampling and Reconstruction

- Continuous Time and Discrete Time - cont.
- Reconstruction
  - Perfect reconstruction cannot be realized
  - Two main sources of error:
    - Aperture effect due to characteristic pulse shape
    - Magnitude and phase errors related to anti-imaging filter
  - If either error is an important parameter, they would need to be measured and corrected


AWG
Waveform Source Memory
DAC
Anti-imaging Filter
DUT
Anti-aliasing Filter
ADC
ATE Digital
Waveform Capture Memory
ATE Digital
Waveform Source Memory
DAC
Anti-imaging Filter
Anti-aliasing Filter
ADC
Waveform Capture Memory
Digitizer

Sampling and Reconstruction

- **Continuous Time and Discrete Time - cont.**
  - In purely mathematical world, there would be no loss of information due to ADC and DAC processes
  - Unfortunately, a number of imperfections from the real world make the conversion between continuous time and discrete time fall short of mathematical theory.
  - Both sampling and reconstruction are used heavily in mixed signal testing, either as part of the DUT or as part of the testing.

Sampling and Reconstruction

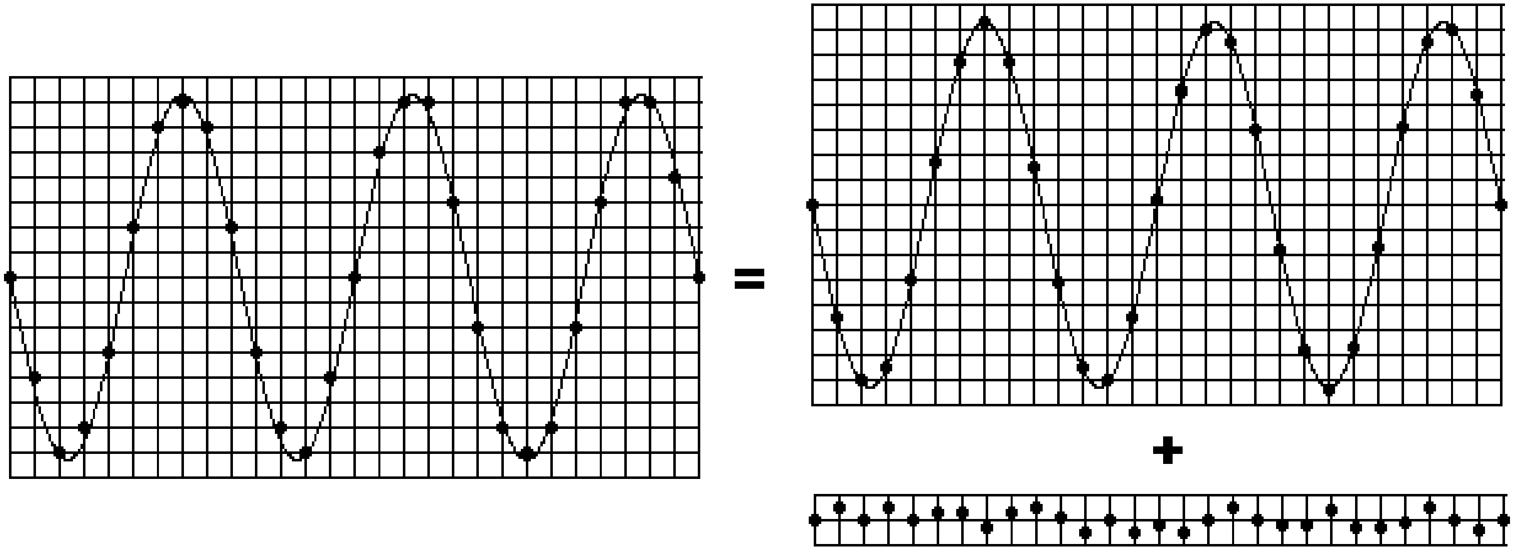
- Sampling
  - Mathematically, a simple program would perform the trick:

```
double sinewave[512], pi=3.1415926
int i;
for (i=0; i<512;i++)
        sinewave[i] = sin(2.0*pi*i/512);
```

  - The only error introduced is in the mathematical error in the computation
  - Q1 What is the period of the above given sin wave?
  - Q2: Why is it always preferred to use 2^n as the number of samples that we collect?

Sampling and Reconstruction

- Sampling
  - Real world sampling with an ADC always introduces some distortion
    - physical noise: Boltzmann Noise, Johnson Noise etc.
    - Conversion noise: Quantization Error
      - Round off to the nearest LSB or Quanta
      - Statistically speaking, Quantization error is uniformly distributed from -1/2 to +1/2 LSB - assuming a perfect ADC
        - What if I do not have perfect ADC? What could happen there?
      - Can be reduced by higher resolution ADC
      - Every added bit of resolution reduces the LSB by one half, and therefore reduces the Quantization Error by a factor two or 6dB.



Quantized Waveform = Ideal Waveform + Quantization Errors

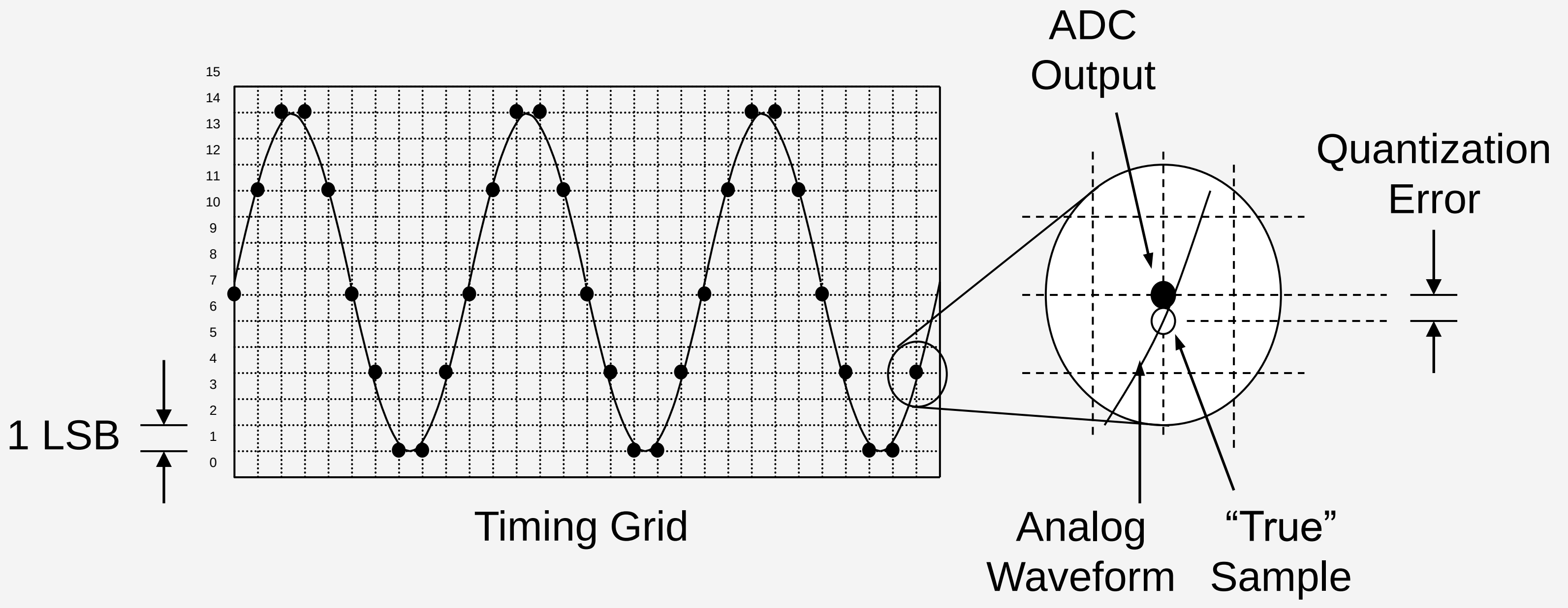
Sampling and Reconstruction

- Sampling
  - Least Significant Bit (LSB) is the distance between the horizontal lines
  - an N bit ADC with a full scale analog input range of F_S has a LSB voltage size of:

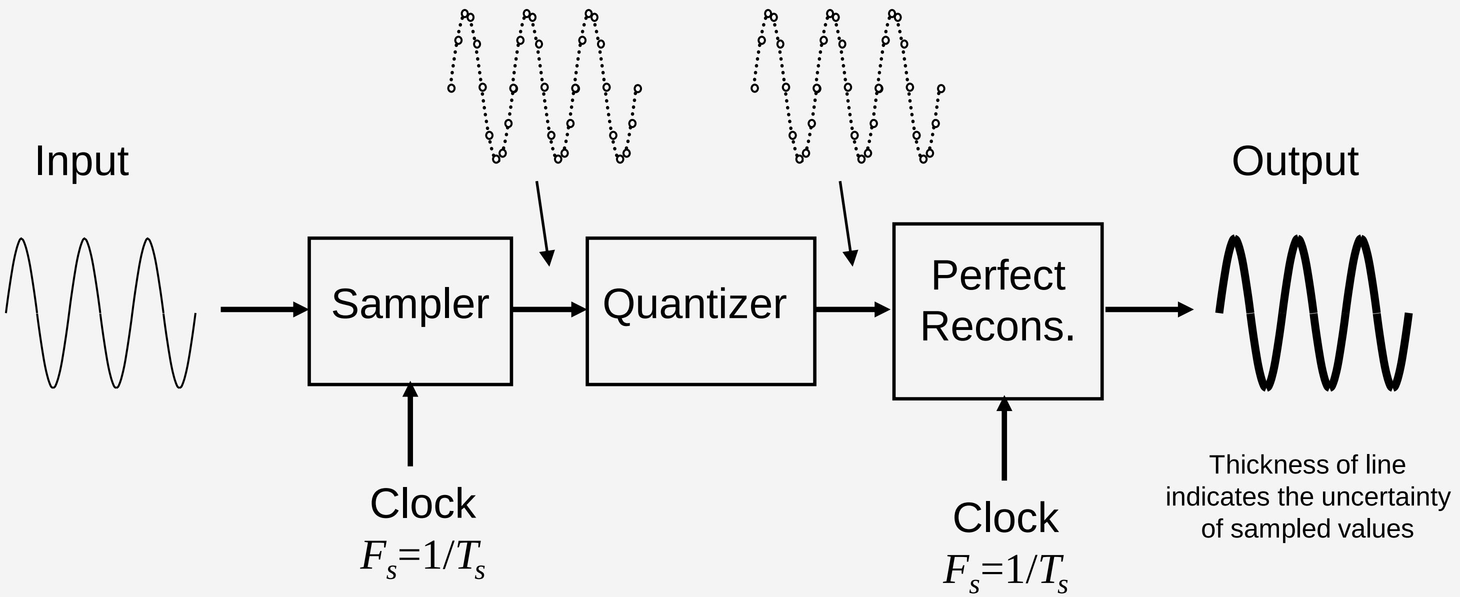
$$V_{LSB} = \frac{F_S}{2^N - 1}$$

  - Quantization errors exhibit a uniform random distribution from -1/2 LSB to 1/2 LSB
  - Perfect sine wave in a perfect ADC no matter how many times, I would get the same error. The above statement assume that the signal also has random distribution error.

Quantization Noise



Quantization Noise cont.



Sampling and Reconstruction

- Reconstruction
  - Mathematical reconstruction is achieved using Fourier Transformations
  - A DAC and a reconstruction filter is used for physical reconstruction of a sampled waveform.
  - Reconstruction filter is also known as an Anti-Imaging Filter. (Low Pass Filter)
  - Reconstruction results in a reduction in gain and a phase shift to the right, so if phase shift is an important parameter, a focused calibration must be performed to correct for the shift.

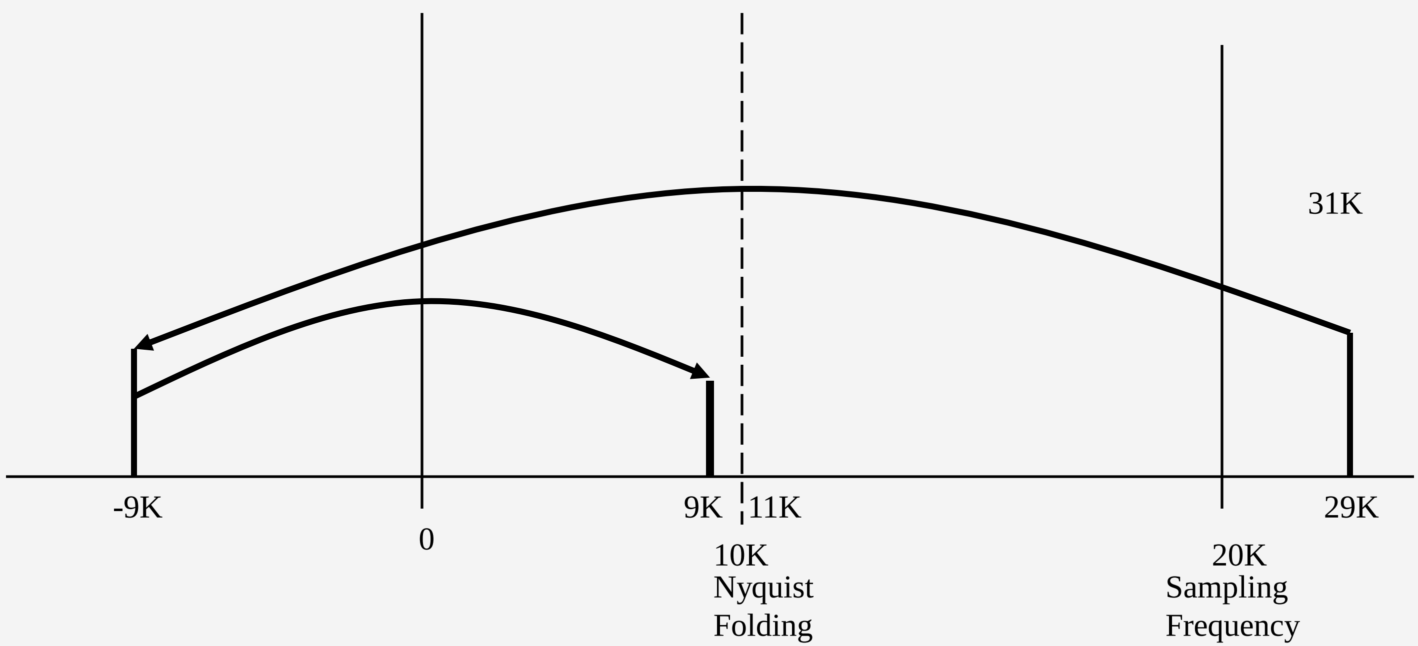
Sampling and Reconstruction

- Aliasing
  - Nyquist Criteria
    - A continuous waveform can be perfectly represented by a limited number of discrete samples, as long as the samples are taken at greater than two times the highest frequency component in the continuous waveform.
    - The sampling rate $2F_{max}$ is called the Nyquist rate and its reciprocal is called the Nyquist interval. The Nyquist rate is the minimum sampling rate allowable by the sampling theorem. Although somewhat confusing at times, the Nyquist frequency refers to Fmax.
    - Aliasing happens when the sampling rate is less than $2F_{max}$

Sampling and Reconstruction*

- Aliasing
  - Nyquist Criteria
    - By changing the number of points sampled per cycle, one can see the effect of aliasing in the time domain.

Aliasing in the Frequency Domain



Let us take a historical detour so we get to know some more about Shannon, his ideas and others in the field…..

This is a basic knowledge for all EE and CprEs

Sampling theory, Shannon, Nyquist, who is who?

- Sampling theory was first discovered by E.Whittaker (in 1915)
  Whittaker E.T., 1915, Proc. Roy. Soc. Edinburgh A. 35, 181
- A Prof of Math at Edinburgh University
- This the origin of the modern Sampling Theory
- It also appeared in Russian literature by Kotel'nikov in 1933

- Shannon understood the engineering implication of this.
  C. E. Shannon, ``A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, pp. 379-423 and 623-656, July and October, 1948.

This is the historical paper that basically defines the information theory.
- While there is historical debate if Shannon knew about the 1915 work, Shannon is basically known as the father of the modern information theory.

So what about Nyquist?

Shannon uses what Nyquist developed to develop his work. It is reference on the first page of his 55 page work

1Nyquist, H., “Certain Factors Affecting Telegraph Speed,” *Bell System Technical Journal*, April 1924, p. 324; “Certain Topics in Telegraph Transmission Theory,” *A.I.E.E. Trans.*, v. 47, April 1928, p. 617.

Shannon and other, what did he say, think do???

While Shannon must get full credit for formalizing this result and for realizing its potential for communication theory and for signal processing, he did not claim it as his own. In fact, just below the theorem, he wrote: "this is a fact which is common knowledge in the communication art." He was also well aware of equivalent forms of the theorem that had appeared in the mathematical literature; in particular, the work of Whittaker [144]. In the Russian literature, this theorem was introduced to communication theory by Kotel'nikov [67], [68].

From: Sampling-50 years after Shannon see the site

Some of the issues that Shannon worked on

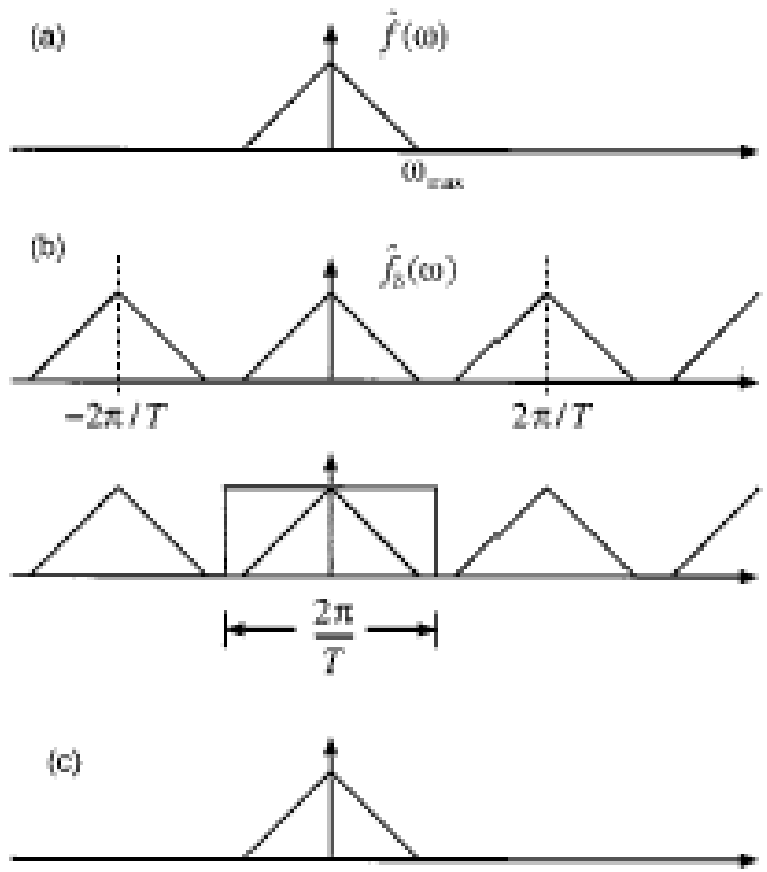
Definition: The capacity C of a discrete channel is given by

$$C = \lim_{T \to \infty} \frac{\log N(T)}{T}$$

where $N(T)$ is the number of allowed signals of duration T.

If a function $f(x)$ contains no frequencies higher than $\omega_{\max}$ (in radians per second), it is completely determined by giving its ordinates at a series of points spaced $T = \pi/\omega_{\max}$ seconds apart.

Fig. 1. Frequency interpretation of the sampling theorem: (a) Fourier transform of the analog input signal $f(x)$, (b) the sampling process results in a periodization of the Fourier transform, and (c) the analog signal is reconstructed by ideal low-pass filtering; a perfect recovery is possible provided that $\omega_{max} \leq \pi/T$.



So, what is Shannon's sampling theorem

If a signal is sampled for all time at a rate more than twice the highest frequency at which CTFT is nonzero, it can be exactly reconstructed from the samples. Signals and Systems   M. J. Roberts

What is CTFT?

Nyquist frequency is the highest frequency fm
The Nyquist rate is 2fm

A signal whose spectrum is band-limited to B Hz [$G(\omega)=0$ for $|\omega| > 2\pi B$] can be reconstructed exactly (without any error) from its samples taken uniformly at a rate of $R > 2B$ Hz (samples per second) IN other word the minimum sampling frequency is $fs=2B$ Modern Digital and Analog Communication Systems B. P. Lathi

Sampling and Reconstruction

- Imaging
  - Imaging falls under the same Nyquist criteria rules as sampling
  - A series of samples can’t be used to reproduce signal components above the Nyquist frequency.
    - If the sampling rate of the DAC is 20kHz, then it can not be used to produce a reconstructed sine wave with a frequency of 12kHz.
  - However, a DAC produces undesirable signal components above the Nyquist frequency called Images.
    - Appear around integer multiples of the sampling frequency.

The Nyquist Sampling Rate

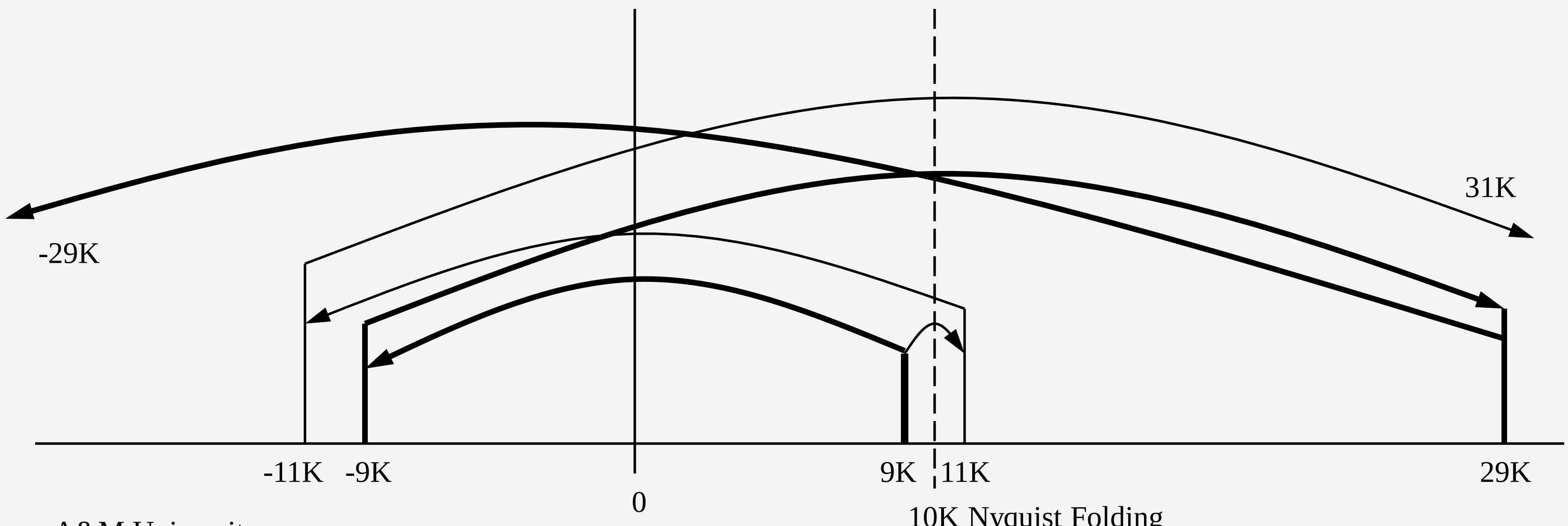
- The Nyquist rate is the maximum signaling rate that is achievable in an ideal low-pass channel with no intersymbol interference

Leon-Garcia Widjaja

Communication Networks

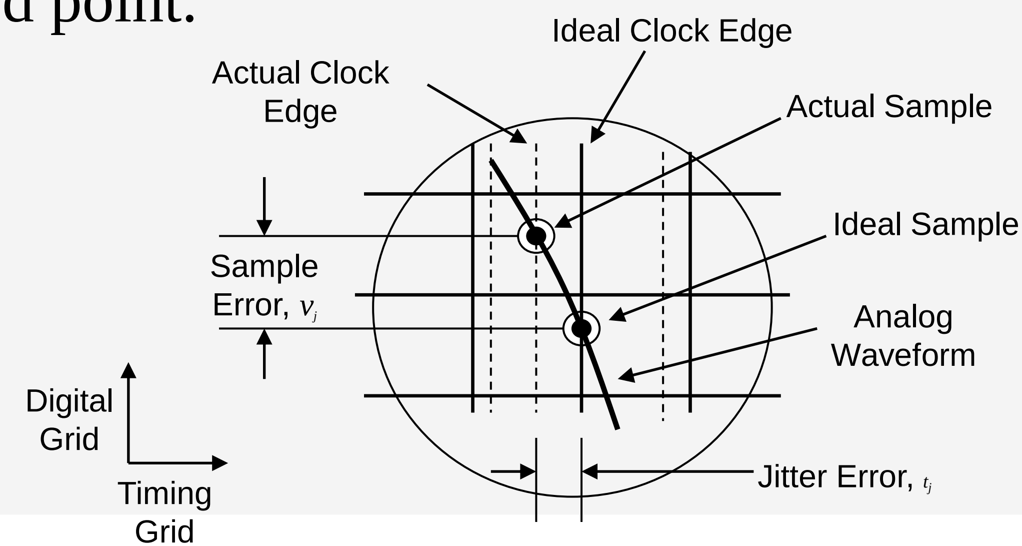
Sampling and Reconstruction

- Imaging - cont.
  - Example: if an unfiltered DAC reconstructs a 9kHz sine wave using a 20kHz sampling rate, then images of the 9kHz signal will appear at 11kHz, 29kHz, 31kHz, 49kHz, 51kHz, and so on.



Sampling and Reconstruction

- Sampling Jitter
  - The error in the placement of each clock edge controlling the sampling point of each ADC or DAC sample.
  - In an ADC the jitter noise is proportional to both the magnitude of the jitter and the slope of the signal at each sampled point.



Sampling and Reconstruction

- Sampling Jitter - cont.
  - Given by the formula: Noise = 2 *pi*A*F*Tj/√2
    where:
    - Noise = RMS noise induced by jitter
    - A = Sine wave amplitude
    - F = Sine wave frequency
    - Tj = Time jitter (RMS seconds)
  - In both DAC or ADC cases, doubling the jitter doubles the noise, doubling the frequency or amplitude doubles the noise.
  - Test engineers job is to find the minimum jitter conditions of the tester and DUT.

Problem - 10 Minutes

- Calculate the RMS noise induced by Jitter in an ADC given an 2.5V RMS Volt input signal at 1000 Hz. Assume a 1 μsec RMS second time jitter.
- What happens to the RMS noise if the frequency is increased by a factor of 10?

Problem - Solution

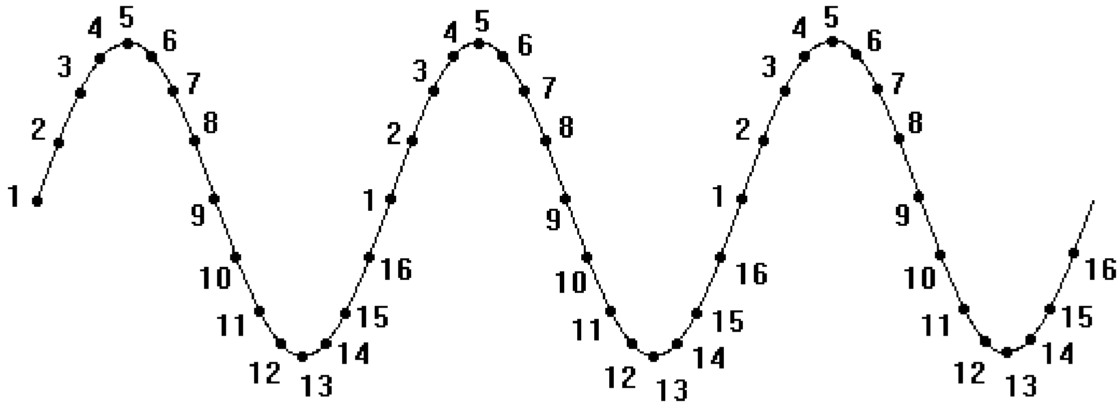
– Jitter Noise = 2 * π * A* F * Tj

- Jitter noise = 0.0157 V RMS
- Increasing the frequency to 10,000 Hz changes the jitter noise to 0.157 V RMS

Repetitive Sampling Sets

- Finite and Infinite Sample Sets
  - Real life signals virtually never repeat themselves
  - Sampled data sets must be considered infinite in order to allow DSP based testing.
  - Signals are created using finite sample sets of a few hundred to thousand data points.
    - These can be repeated endlessly
    - In production testing, repeat time causes long test times, so repetition is limited to the minimum required to gather response data
  - It is critically important that the last data point flow smoothly into the first data point of the repeating sequence - called coherence

A fine sample set will be easy to handle and predicted



What happens if we get the above sin wave, and sample it in the 1-16 points, repeatedly?

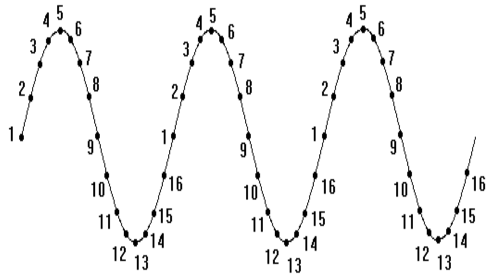
The fundamental frequency will be $F_s/16$ for

N samples collected at the rate of Fs the fundamental frequency is $F_f=F_s/N$ also known as frequency resolution

Then $1/F_f$ is the fundamental period (UTP, Unit Test Period)

The amount of time to get N samples at the rate of F_s is N/F_s

Example



If source the sample at 16kHz then the fundamental frequency is 16k/16=1kHz

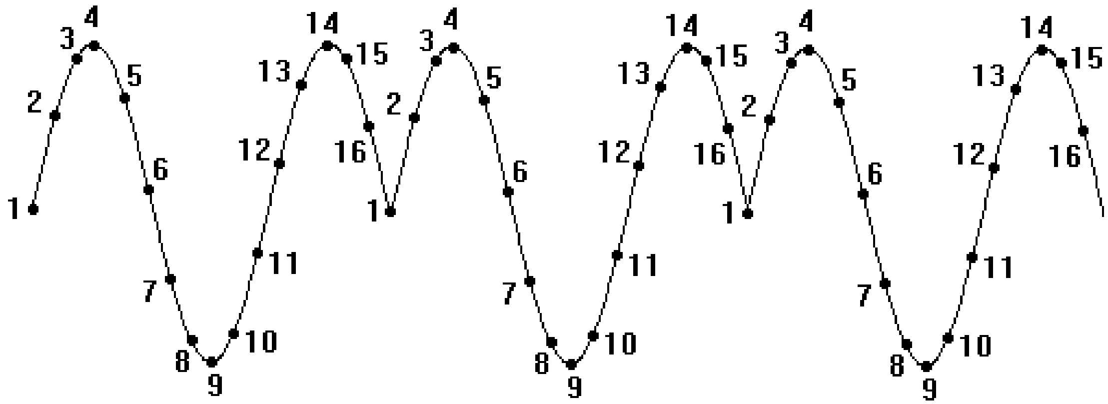
With the 1kHz fundamental frequency the sine wave appears at 1kHz. With the same 16 samples we could also produce 2kHz. But 1.5 2.5 etc cannot be produced. In order to produce 1.5 KHz since wave, we need to use coherent sample rate.

So what would we do?

We need to choose a sampling system with 1.5kHz/N fundamental frequency (N is an integer). If we choose 500Hz and then use the third multiple of the fundamental, we can produce 1.5 kHz

How do we do 500 Hz fundamental frequency if we do 16kHz samples? 16kHz/32=500Hz.

Fundamental frequency determines the Frequency resolution of a measurement. Why should we not minimize the Ff? TEST TIME CONSTRAINT would be the issue



Non coherent Sample sets cannot be loaded properly

Repetitive Sampling Sets

- Coherent and Non-Coherent Signals
  - Reconstruction of the sample set at a certain sampling frequency results in a wave with a frequency of sampling frequency/number of samples. This is called the Fourier frequency, fundamental or primitive frequency. Primitive period or unit test period (UTP) of the signal = 1/primitive frequency.
  - The amount of time required to collect a set of samples is equal to the UTP of the signal being sampled.
  - In practice it takes an extra fraction of UTP to allow DUT and ATE hardware to settle.

Repetitive Sampling Sets

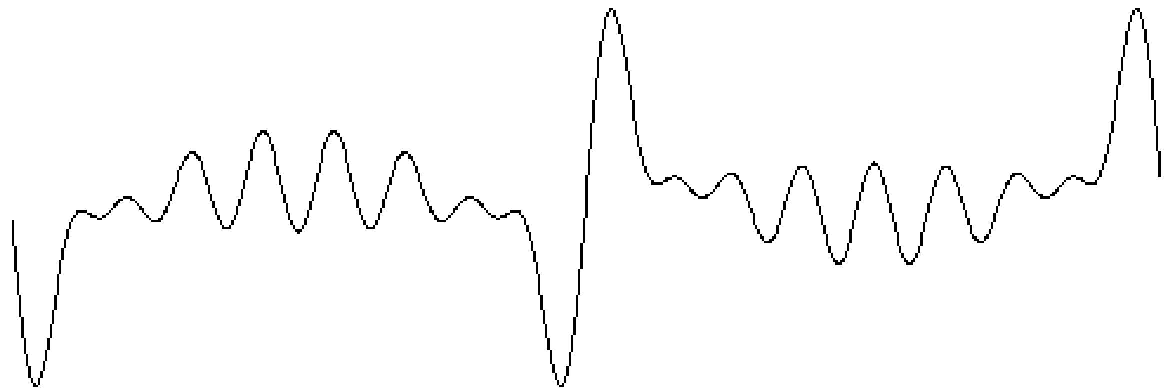
- Coherent and Non-Coherent Signals cont.
  - The Fourier frequency of the signal is called the frequency resolution (since only multiples of the Fourier frequency can be produced coherently with sampled data sets i.e. F_f, $2*F_f$, $3*F_f$, etc. up to the Nyquist frequency)
  - Since the Fourier frequency determines the resolution, a minimum Fourier frequency would allow a maximum of signals that can be sourced.
  - Unfortunately, the lower the Fourier frequency, the longer the test time, which is undesirable in production testing.

Repetitive Sampling Sets

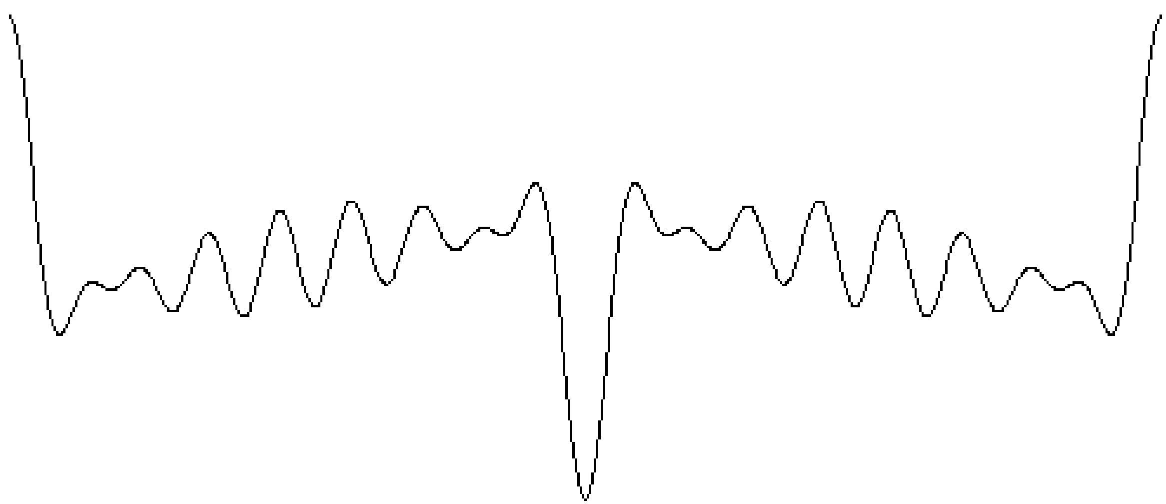
- Coherent and Non-Coherent Signals cont.
  - Any signal made up of a sum of coherent signals is also coherent.
  - If one or more of the frequency components are non-coherent the entire waveform will be non-coherent.
  - Non-coherent sample sets can be analyzed with windowing operations. (Chapter 7)
  - Non-coherent windowing is an inferior measurement technique than coherent non-windowing.

Repetitive Sampling Sets

- Peak to RMS Control in Coherent Monotones
  - If all multi-tone signals are created with zero phase shift, summation of the frequencies causes large spikes in the input waveform, which could lead to saturation of your circuit.
  - Randomized phase shifting is a common practice in mixed signal testing, once the phases are determined, they are hard coded into the test program to avoid correlation problems.
  - There is no standard formula to determine the optimum phase shifts for a multi-tone signal.



Equal Amplitude Sine Waves



Equal Amplitude Cosine Waves

Repetitive Sampling Sets

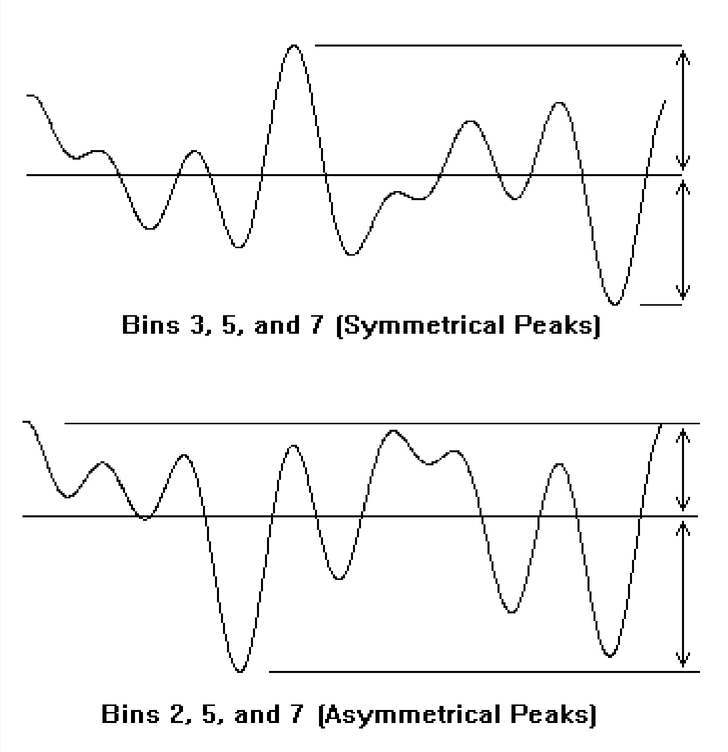
- Spectral Bin Selection
  - Spectral bins are not the same bins as used in selecting the quality of the DUT. This is a reference to the multiplication factor of the Fourier frequency which results in the combination of a multi-tone.
  - The calculated bin may not be mutually prime with the number of samples, so a shift may be necessary.
    - Non mutually prime bin numbers results in repetitively sampling the same voltage points in a waveform and periodic errors from quantization.
    - Also, using even bins causes the DC offset of your multi-tone to be shifted positively, so the DUT is not exercised adequately in the negative voltage ranges.

Repetitive Sampling Sets

- Spectral Bin Selection
  - So, you must: use mutually prime bin numbers, use odd bin numbers and if all else fails, use even bin numbers
  - Also, watch that there is no overlap in harmonic or intermodulation frequencies with your test tone bins
- Multitone DSP based testing only works if care is taken in selecting the test tones.
  - All harmonics and intermodulation frequencies must be taken into account
  - Multi-tone coherence must be maintained.

Repetitive Sampling Sets

- Spectral Bin Selection
  - A combination of odd and even bins would have yielded an asymmetrical positive and negative peaks with respect to DC offset of the signal.



Problem - 15 Minutes

- Select the spectral bins and actual test tone frequencies for a five tone multi-tone at 500 Hz, 1.5kHz, 2kHz, 2.5kHz, and 3kHz with no more than 100 Hz error in the signal frequencies. The signal should take no more than 100 msec. To repeat. Use a 32kHz sampling rate.

Problem - Solution

- The 100 Hz error limit means the Fourier frequency can be no more than 200Hz. In order to get prime bins we must divide by at least the number of frequencies +1 (6) = thus the Maximum Fourier frequency is 33.33 Hz
- The 100 msec. repeat limit means the Fourier frequency must be greater than 1/UTP = 1/100msec = thus the minimum Fourier frequency is 10 Hz.
- Maximum number of samples = F_S/F_{fmin} = 32kHz/10Hz = 3,200 samples
- Minimum number of samples = F_S/F_{fmax} = 32kHz/33.33Hz = 960 samples
- Leaves two choices with power of two 1024 and 2048
- Choose 1024 to minimize test time.
- So the Fourier Frequency we use is 32kHz/1024 = 31.25Hz

Problem - Solution cont.

- 500Hz bin = 500Hz/31.25 = 16 (non-prime shift to 17)
  - Test frequency = 17 * Ff = 531.25, error = 31.25Hz
- 1.5kHz bin = 1500Hz/31.25 = 48 (non-prime shift to 47)
  - Test frequency = 47 * Ff = 1468.75, error = 31.25Hz
- 2kHz bin = 2000Hz/31.25 = 64 (non-prime shift to 67)
  - Test frequency = 67 * Ff = 2093.75, error = 93.75Hz
- 2.5kHz bin = 2500Hz/31.25 = 80 (non-prime shift to 79)
  - Test frequency = 79 * Ff = 2468.75, error = 31.25Hz
- 3kHz bin = 3000Hz/31.25 = 96 (non-prime shift to 97)
  - Test frequency = 97 * Ff = 3031.25, error = 31.25Hz

Problem - 10 Minutes

- Verify that the harmonics and 2nd order Intermodulation terms do not interfere with the test tones in the previous example. Assume an anti-aliasing filter with a cutoff frequency of 16kHz.

Problem - Solution

- Since the anti-aliasing filter will remove any frequency component above bin 512, we need only concern ourselves with bins 1-512. (actually only to bin 100 since our test tones are limited to this frequency range)
- Harmonics of bin 17 = 34, 51, 68, 85, 102
- Harmonics of bin 47 = 94, 141
- Harmonic of bin 67 = 134
- Harmonic of bin 79 = 158
- Harmonic of bin 97 = 194
- Second order (F1 - F2) Intermodulation frequencies = 30, 50, 62, 80, 20, 32, 12, 18
- Second order (F1 + F2) Intermodulation frequencies = 64, 84, 114, 96, 126, 146, 144, 164, 176
- Third order Intermodulation frequencies are 2*F1 ± F2 and F1 ± 2*F2

Synchronization of Sampling Systems

- Multiple Sampling Systems in Simultaneous Testing
  - The simplest way to insure coherence in synchronized testing is by setting all Fourier frequencies of all sampling systems to the same value.
  - Unfortunately, all ATE testers have a limitation on the sampling rates available, i.e. you can sample at any frequency as long as it is a multiple of 4 Hz.

Synchronization of Sampling Systems

- ATE Clock Sources
  - Ultimately all clocks in a mixed signal tester should be referenced to a single master clock so that all instrumentation can be used for coherent DSP based testing.
    - Phase Locked Loops - produces an output clock equal to a reference clock times M divided by N.
    - Frequency Synthesizers - uses a reference clock and passes it through a series of frequency mixers to produce a stable output frequency.
    - Flying Adders - uses a high speed reference clock and the timing signal through an adder circuit to calculate the delay on the fly.

Synchronization of Sampling Systems

- The Challenge of Synchronization
  - The clock signals are often modified further by instruments or subsystems that use them.
  - Clearly setting the sampling rates in an ATE tester is one of the more difficult tasks of a test engineer, but through careful attention to details and systematic tracking of constrains the test engineer can be successful in coherent DSP based Testing

Synchronization of Sampling Systems

- The Challenge of Synchronization
  - It can be a maddening process to calculate a coherent sampling system without violating any of the tester's clocking rules. The digital pattern must run at a particular frequency, specified in the DUT's data sheet. The DUT's DAC and ADC must run at particular frequencies. The tester's clocks must fall within certain ranges at each stage in the frequency divider chains. Sample sizes should be powers of two for use of efficient FFT routines. Constant reprogramming of the master clock may add extra test time because of frequency synthesizers. Finally, all the sampling rates and number of points must result in the same fundamental frequency for all the DACs and ADCs in the DUT and tester.