EE 505
Lecture 11

• Statistical Circuit Modeling

R-string Example
Offset Voltages
Current Steering DAC Statistical Characterization

Binary Weighted

\[
\sigma_{\text{INL}_{b=1000\ldots0}} = \sqrt{\frac{N}{2} \left[ 1 - \frac{N}{2} \frac{N}{N-1} \right]^2 + \left( \frac{N}{2} - 1 \right) \left( \frac{N}{2} \frac{N}{N-1} \right)^2 \cdot \sigma_{\frac{I_{RGK}}{I_{LSBX}}} ^2}
\]

\[
\sigma_{\text{INL}_{\text{MAX}}} \approx \sigma_{\text{INL}_{b=1,0\ldots0}} \approx \sqrt{\frac{N}{2}} \sigma_{\frac{I_{RG}}{I_{LSBX}}} ^2
\]

Note this is the same result as obtained for the unary DAC

But closed form expressions do not exist for the INL of this DAC since the INL is an order statistic
Statistical Modeling of Current Sources

Simple Square-Law MOSFET Model Usually Adequate for static Statistical Modeling

Assumption: Layout used to eliminate gradient effects, contact resistance and drain/source resistance neglected

\[ I_D = \frac{\mu C_{OX} W}{2L} (V_{GS} - V_{TH})^2 \]

Random Variables: \( \mu, C_{OX}, V_{TH}, W, L \)  
Thus \( I_D \) is a random variable

Need: \[ \frac{\sigma_{I_D}}{I_{DN}} \]
Review from previous lecture:

**Statistical Modeling of Current Sources**

\[ \sigma_{l_{DR}} = \sqrt{\sigma_{\frac{\mu_R}{\mu_N}}^2 + \sigma_{\frac{C_{OXR}}{C_{Oxn}}}^2 + 4 \left( \frac{V_{THN}}{V_{GS} - V_{THN}} \right)^2 \sigma_{\frac{2}{V_{THN}}}^2} \]

or

\[ \sigma_{l_{DR}} = \sqrt{\sigma_{\frac{\mu_R}{\mu_N}}^2 + \sigma_{\frac{C_{OXR}}{C_{Oxn}}}^2 + \left( \frac{2}{V_{GS} - V_{THN}} \right)^2 \sigma_{V_{THN}}^2} \]

It will be argued that

\[ \sigma_{\frac{\mu_R}{\mu_N}}^2 = \frac{A_{\mu}^2}{WL} \]

\[ \sigma_{\frac{C_{OXR}}{C_{Oxn}}}^2 = \frac{A_{Cox}^2}{WL} \]

\[ \sigma_{V_{THR}}^2 = \frac{A_{VT0}^2}{WL} \]

\[ A_{\mu}, A_{Cox}, A_{VT0} \] are process parameters

\[ \sigma_{l_{DR}} = \frac{1}{\sqrt{WL}} \sqrt{A_{\mu}^2 + A_{Cox}^2 + \frac{4}{V_{EB}^2} A_{VT0}^2} \]

Define

\[ A_\beta = \sqrt{A_{\mu}^2 + A_{Cox}^2} \]

Thus

\[ \sigma_{l_{DR}} = \frac{1}{\sqrt{WL}} \sqrt{A_\beta^2 + \frac{4}{V_{EB}^2} A_{VT0}^2} \]

Often only \( A_\beta \) is available
Statistical Modeling of Current Sources

\[ \sigma_{I_{\text{DR}} I_{\text{DN}}} = \frac{1}{\sqrt{WL}} \sqrt{A_\beta^2 + \frac{4}{V_{EB}^2} A_{VT0}^2} \]

- Standard Deviation Decreases with Sqrt of A
- Large \( V_{EB} \) reduces standard deviation
- Operating near cutoff results in large mismatch
- Often threshold voltage variations dominate mismatch

\[ \sigma_{I_{\text{DR}} I_{\text{DN}}} \approx \frac{2}{V_{EB} \sqrt{WL}} A_{VT0} \]
Statistical Modeling of Circuits

The previous statistical analysis was somewhat tedious

Will try to formalize the process

Assume $Y$ is a function of $n$ random variables $x_{R1}, \ldots x_{Rn}$ where the mean and variance of $x_{Ri}$ are “small”

$$Y = f \left( x_{R1}, x_{R2}, \ldots x_{Rn} \right)$$

$$X_R = \begin{bmatrix} x_{R1} \\ x_{R2} \\ \vdots \\ x_{Rn} \end{bmatrix}$$

$$Y = f \left( X \right) \bigg|_{X_R = 0} + \sum_{j=1}^{n} \frac{\partial f}{\partial x_{Rj}} \bigg|_{X_R = 0} x_{Rj} + \varepsilon \left( x_{R1}, x_{R2}, \ldots x_{Rn} \right)$$

where $\varepsilon \left( x_{R1}, x_{R2}, \ldots x_{Rn} \right)$ small
Review from previous lecture:

**Statistical Modeling of Circuits**

Alternatively, if we define

\[
S_{xRj}^f = \frac{x_{iN}}{Y_N} \frac{\partial f}{\partial x_{Rj}} \bigg|_{x_R=0}
\]

we thus obtain

\[
\sigma^2_Y = \sum_{j=1}^{n} \left( \left[ S_{xRj}^f \right]^2 \sigma^2_{x_{Rj}} \right) \frac{x_{Ni}}{x_{Ni}}
\]
Review from previous lecture:

Thm:

If $I_1$ and $I_2$ are two current sources, $\Delta I = I_1 - I_2$

\[ \sigma_{\Delta I} = \sqrt{2} \sigma \frac{I_R}{I_N} \]

rf:

$\Delta I = I_1 - I_2$

\[ \frac{\Delta I}{I_N} = \frac{I_1}{I_N} - \frac{I_2}{I_N} \]

\[ = \frac{I_N + I_{R1}}{I_N} - \left( \frac{I_N + I_{R2}}{I_N} \right) \]

\[ \Delta \frac{I}{I_N} = \frac{I_{R1}}{I_N} - \frac{I_{R2}}{I_N} \]

\[ \therefore \sigma_{\frac{\Delta I}{I_N}} = \sigma^2 \frac{I_{R1}}{I_N} + \sigma^2 \frac{I_{R2}}{I_N} = 2\sigma^2 \frac{I_R}{I_N} \]
Often simulations are used to predict statistical performance of a circuit.

Note: \( I_R, R_R, C_R, V_{osR}, \ldots \) are often Gaussian.

Most CAD tools do not have a rich set of random variable distributions. Most CAD tools have only a \( U[0,1] \) random number generator.
Review from previous lecture:

Theorem: If \( x \sim \mathcal{U}[0,1] \), then \( y = F^{-1}(x) \) has a pdf of \( f(y) \) where \( f(y), F(y) \) are any pdf/cdf pair.

Corollary: If \( x \sim \mathcal{U}[0,1] \) and

\[
F(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{h} e^{-\frac{x^2}{2}} \, dx,
\]

then \( y = F^{-1}(x) \) is \( \mathcal{N}(0,1) \).

CDF showing random variable mapping of \( x_1 \) from \( \mathcal{U}(0,1) \).
Example: Determine the area required for the resistors for an n-bit R-string DAC to achieve a yield of P if the device is marketable provided \[ |INL_{k\text{MAX}}| < \frac{1}{2} \text{LSB} \]

Solution: \( INL_{k\text{MAX}} \) is Gaussian

Assume the area of each resistor is WL

\[ \mu = 0 \]
\[ \sigma = \sqrt{\frac{N}{2}} \sqrt{\frac{R_a}{R_N}} \]

but \( \sigma_{R_a}^2 = \frac{A_p^2}{W \cdot L \cdot \rho_{\text{Nom}}^2} \)

\[ A_R = \frac{A_p}{\rho_{\text{NOM}}} \]

\[ \sigma_{\frac{R_a}{R_N}} = \frac{A_R}{\sqrt{WL}} \]
Since \( P \) is fixed, can solve for \( X_n \)
\[ P = 2F_N\left( X_N \right) - 1 \quad \Rightarrow \quad X_N = F_N^{-1}\left( \frac{P+1}{2} \right) \]

where \( F_N(X_N) \) is the CDF on \( N(0,1) \)

\[ X_N = \frac{x}{\sigma} = \frac{1}{2} \frac{2}{\sqrt{N}} \sigma_{\frac{R_R}{R_N}} = \frac{1}{\sqrt{N}} \sigma_{\frac{R_R}{R_N}} \]

\[ \sigma_{\frac{R_R}{R_N}} = \frac{A_R}{\sqrt{WL}} \]

\[ X_N\bigg|_{x=\frac{1}{2}} = \frac{\sqrt{WL}}{A_R \sqrt{N}} \]

\[ \sqrt{WL} = A_R \sqrt{N} X_N \quad \Rightarrow \quad \sqrt{WL} = A_R \sqrt{N} \cdot F_N^{-1}\left( \frac{P+1}{2} \right) \]
\[
\sqrt{WL} = A_R \sqrt{N} X_N \quad \Rightarrow \quad \sqrt{WL} = A_R \sqrt{N} \cdot F_N^{-1}\left(\frac{P+1}{2}\right)
\]

Total area

\[
A_{TOT} = 2^n \sqrt{WL} = 2^n A_R \sqrt{N} X_N = 2^n A_R 2^{\frac{n}{2}} X_N = \sqrt{2^{3n}} A_R X_N
\]

or equivalently

\[
A_{TOT} = \sqrt{2^{3n}} A_R X_N
\]
Offset Voltages

All ADCs have comparators and many ADCs and DACs have operational amplifiers.

The offset voltages of both amplifiers and comparators are random variables and invariably are key factors affecting the performance of a data converter.

Operational Amplifiers:

- Generally differential amplifiers whose offset is dominantly determined by randomness in the first stage.

Comparators:

- High Gain Operational Amplifiers
- Latching Structures (often clocked)
- Combination of High Gain Amplifiers and Latching Structures

- Offset voltages of high-gain amplifiers well understood
- Offset voltage of Latching Structures often difficult to determine and can be very large
Consider First Offset in Operational Amplifiers

Input-referred Offset Voltage: Differential Voltage that must be applied to the input to make the output assume its desired value

Note: With a good design, a designer will have $V_{\text{OUT}}$ at the desired value if the components assume the values used in the design.

Any difference in the output from what is desired when components assume the nominal values used in a design is attributable to a systematic offset voltage.
Offset Voltage in Op Amps

Assume desired currents at output must satisfy \( I_4 = I_2 \)

Strategy:
1) Obtain an expression for \( V_{os} \)
2) Linearize this expression and decorrelate
3) Obtain \( \Delta V_{os} \)
Analysis of Offset Voltage

\[ I_{D1} = \frac{\mu_1 C_{OX1} W_1}{2 L_1} (V_1 - V_S - V_{T1})^2 \]  \hspace{1cm} (1)

\[ I_{D2} = \frac{\mu_2 C_{OX2} W_2}{2 L_2} (V_2 - V_S - V_{T2})^2 \]  \hspace{1cm} (2)

\[ I_{D3} = \frac{\mu_3 C_{OX3} W_3}{2 L_3} (V_X - V_{DD} - V_{T3})^2 \]  \hspace{1cm} (3)

\[ I_{D4} = \frac{\mu_4 C_{OX4} W_4}{2 L_4} (V_X - V_{DD} - V_{T4})^2 \]  \hspace{1cm} (4)

Since \( \sqrt{I_{D1}} = \sqrt{I_{D3}} \)

\[ V_1 - V_S - V_{T1} = \sqrt{\frac{L_1 \mu_3 C_{OX3} W_3}{L_3 \mu_m C_{OX1} W_1}} (V_X - V_{DD} - V_{T3}) \]

Since \( \sqrt{I_{D2}} = \sqrt{I_{D4}} \)

\[ V_2 - V_S - V_{Tn2} = \sqrt{\frac{L_2 \mu_4 C_{OX4} W_4}{L_4 \mu_n C_{OX2} W_2}} (V_X - V_{DD} - V_{Tn4}) \]
Analysis of Offset Voltage

Define: 
\[ a = \frac{L_1 \mu_{p3} C_{OX3} W_3}{L_3 \mu_{n1} C_{OX1} W_1} \]  
\[ b = \frac{L_2 \mu_{p4} C_{OX4} W_4}{L_4 \mu_{n2} C_{OX2} W_2} \]

Substituting for \( a \) and \( b \), it follows that
\[ V_{OS} = V_2 - V_1 = V_{tn2} - V_{tn1} + (b - a)(V_X - V_{DD}) + aV_{tp3} - bV_{tp4} \]

Assume 
\[ V_X = V_{XN} - V_{XR} \]
\[ a = a_N + a_R \]
\[ b = b_N + b_R \]
\[ V_{tni} = V_{tnN} + V_{tnRi} \quad i = 1, 2 \]
\[ V_{tpi} = V_{tpN} + V_{tpRi} \quad i = 3, 4 \]

Observe \( a_N = b_N \) and \( V_{XN} - V_{DD} - V_{TpN} = V_{EB3} \)

It follows that
\[ V_{OS} = V_{tnR2} - V_{tnR2} + (b_R - a_R)V_{EB3} + a_N(V_{tpR3} - V_{tpR4}) \]
\[ \sigma_{V_{OS}}^2 = 2\sigma_{V_{tnR2}}^2 + a_N^2 2\sigma_{V_{tpR3}}^2 + V_{EB3}^2 \sigma_{a_R-b_R}^2 \]

Will now obtain \( a_R \) and \( b_R \)
Analysis of Offset Voltage

\[ V_{OS} = V_{TnR2} - V_{TnR2} + (b_R - a_R) V_{EB3} + a_N \left( V_{TpR3} - V_{TpR4} \right) \]

\[ a = \sqrt{\frac{(L_{N1} + L_{R1})(\mu_{Np3} + \mu_{R3})(C_{OXN3} + C_{OXR3})(W_{N3} + W_{R3})}{(L_{N3} + L_{R3})(\mu_{Nn1} + \mu_{R1})(C_{OXN1} + C_{OXR1})(W_{N1} + W_{R1})}} \]

Recall for \( x \) small,

\[ \sqrt{1 + x} \approx 1 + \frac{x}{2} \quad \frac{1}{1+x} \approx 1 - x \]

\[ a = \sqrt{\frac{(L_{N1} \mu_{Np3} W_{N3})}{(L_{N3} \mu_{Nn1} W_{N1})}} \left( 1 + \frac{1}{2} \left[ \frac{L_{R1} - L_{R3}}{L_{N1} L_{N3}} + \frac{\mu_{R3} - \mu_{R1}}{\mu_{Np3} \mu_{Nn1}} + \frac{C_{OXR3} - C_{OXR1}}{C_{OXN3} C_{OXN1}} + \frac{W_{R3} - W_{R3}}{W_{N3} W_{N3}} \right] \right) \]

Thus

\[ a_R = \sqrt{\frac{(L_{N1} \mu_{Np3} W_{N3})}{(L_{N3} \mu_{Nn1} W_{N1})}} \frac{1}{2} \left[ \frac{L_{R1} - L_{R3}}{L_{N1} L_{N3}} + \frac{\mu_{R3} - \mu_{R1}}{\mu_{Np3} \mu_{Nn1}} + \frac{C_{OXR3} - C_{OXR1}}{C_{OXN3} C_{OXN1}} + \frac{W_{R3} - W_{R3}}{W_{N3} W_{N3}} \right] \]

\[ a_N = \sqrt{\frac{(L_{N1} \mu_{Np3} W_{N3})}{(L_{N3} \mu_{Nn1} W_{N1})}} \]

Likewise

\[ b_R = \sqrt{\frac{(L_{N1} \mu_{Np3} W_{N3})}{(L_{N3} \mu_{Nn1} W_{N1})}} \frac{1}{2} \left[ \frac{L_{R2} - L_{R4}}{L_{N2} L_{N4}} + \frac{\mu_{R4} - \mu_{R2}}{\mu_{Np4} \mu_{Nn2}} + \frac{C_{OXR4} - C_{OXR2}}{C_{OXN4} C_{OXN2}} + \frac{W_{R4} - W_{R2}}{W_{N4} W_{N2}} \right] \]
Analysis of Offset Voltage

\[ a_R - b_R = \sqrt{\left( \frac{L_{N1} \mu_{Np3} W_{N3}}{L_{N3} \mu_{Nn1} W_{N1}} \right)} \left( \frac{L_{R1} - L_{R2} + L_{R4} - L_{R3} + \mu_{R3} - \mu_{R4} + \mu_{R2} - \mu_{R1}}{L_{N1} L_{N2} L_{N4} L_{N3} \mu_{Np3} \mu_{Np4} \mu_{Nn2} \mu_{Nn1}} \right) \]

\[ \sigma^2_{a_R - b_R} = \left( \frac{L_{N1} \mu_{Np3} W_{N3}}{L_{N3} \mu_{Nn1} W_{N1}} \right) \left[ \frac{\sigma^2_{R1}}{L_{N1} L_{N3}} + \frac{\sigma^2_{R3}}{L_{N3}} + \frac{\sigma^2_{R3}}{\mu_{Np3}} + \frac{\sigma^2_{R2}}{\mu_{Nn2}} + \frac{\sigma^2_{C_{OXR3}}}{C_{OXR3}} + \frac{\sigma^2_{C_{OXR1}}}{C_{OXR1}} + \frac{\sigma^2_{W_{R3}}}{W_{N3}} + \frac{\sigma^2_{W_{R1}}}{W_{N1}} \right] \]

Thus

\[ \sigma^2_{V_{OS}} = 2 \sigma^2_{V_{trR2}} + 2 \frac{L_{N1} \mu_{Np3} W_{N3}}{L_{N3} \mu_{Nn1} W_{N1}} \sigma^2_{V_{trR3}} \]

\[ + \epsilon_{EB3} \left( \frac{L_{N1} \mu_{Np3} W_{N3}}{L_{N3} \mu_{Nn1} W_{N1}} \right)^2 \left[ \frac{\sigma^2_{R1}}{L_{N1} L_{N3}} + \frac{\sigma^2_{R3}}{L_{N3}} + \frac{\sigma^2_{R3}}{\mu_{Np3}} + \frac{\sigma^2_{R2}}{\mu_{Nn2}} + \frac{\sigma^2_{C_{OXR3}}}{C_{OXR3}} + \frac{\sigma^2_{C_{OXR1}}}{C_{OXR1}} + \frac{\sigma^2_{W_{R3}}}{W_{N3}} + \frac{\sigma^2_{W_{R1}}}{W_{N1}} \right] \]
Analysis of Offset Voltage

but

\[ \sigma_{Vr}^2 = \frac{A_{VT0}^2}{WL}, \quad \sigma_{\mu_R}^2 = \frac{A_{\mu}^2}{WL}, \quad \sigma_{\text{Coxr}}^2 = \frac{A_{\text{Cox}}^2}{WL}, \quad \sigma_{L_R}^2 = \frac{2A_L^2}{WL^2}, \quad \sigma_{W_R}^2 = \frac{2A_W^2}{W^2L} \]

So the offset variance can be expressed as

\[ \sigma_{V_{os}}^2 = 2 \frac{A_{VTn0}^2}{W_1L_1} + 2 \frac{\mu_p L_1}{\mu_n W_1} \frac{A_{VTp0}^2}{L_2^2} + V_{EB3}^2 \frac{\mu_p L_1 W_3}{\mu_n L_3 W_1} \frac{1}{2} \left[ \frac{A_{\mu_n}^2}{W_3 L_3} + \frac{A_{\mu_p}^2}{W_1 L_1} + A_{\text{Cox}}^2 \left( \frac{1}{W_3 L_3} + \frac{1}{W_1 L_1} \right) + A_W^2 \left( \frac{2}{W_3^2 L_3} + \frac{2}{W_1^2 L_1} \right) + A_L^2 \left( \frac{2}{W_1^2 L_1} + \frac{2}{W_3^2 L_3} \right) \right] \]

Often this can be approximated by

\[ \sigma_{V_{os}}^2 = 2 \frac{A_{VTn0}^2}{W_1L_1} + 2 \frac{\mu_p L_1}{\mu_n W_1} \frac{A_{VTp0}^2}{L_2^2} + V_{EB3}^2 \frac{\mu_p L_1 W_3}{\mu_n L_3 W_1} \frac{1}{2} \left[ \frac{A_{\mu_n}^2}{W_3 L_3} + \frac{A_{\mu_p}^2}{W_1 L_1} + A_{\text{Cox}}^2 \left( \frac{1}{W_3 L_3} + \frac{1}{W_1 L_1} \right) \right] \]

Or even approximated by

\[ \sigma_{V_{os}}^2 = 2 \frac{A_{VTn0}^2}{W_1L_1} + 2 \frac{\mu_p L_1}{\mu_n W_1} \frac{A_{VTp0}^2}{L_2^2} \]
End of Lecture 11
Statistical Characterization of Resistors – some additional issue

From previous model:

$$\sigma_{\frac{R_R}{R_{NOM}}} = \frac{A_R}{\sqrt{WL}}$$

Note: Have neglected

- edge roughness,
- contact resistance
- interconnect resistance