Pole-zero pairing strategies for cascaded switched-capacitor filters

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Abstract: The effect of pole-zero pairing on the total capacitance of cascaded biquad SC filters is investigated. It is shown that significant reductions in total capacitance and hence corresponding reductions in silicon area are possible through optimal pole-zero pairing without causing significant degradations in filter performance.

1 Introduction

Switched-capacitor (SC) filter design methods have reached a certain level of maturity [1–2]. Specific SC biquadratic structures suitable for cascaded biquad designs have evolved [1–8] which are now widely used because of the recognised low sensitivity characteristics of these structures relative to those attainable with existing alternative schemes. Methods which allow designers to compute the capacitor ratios for these structures rapidly and exactly, to realise a predetermined set of design specifications are also available. Several groups have actually created a ‘SC silicon compiler’ with programs which automatically select a good filter structure based upon the given design specifications, size all components, and generate a circuit layout which can be submitted for fabrication. The problems of pole-zero pairing and the minimisation of total capacitance still prevail.

Pole-zero pairing has been discussed in active RC filter structures for optimal dynamic range and inband losses [9–11]. For the case of SC filter implementations, the pole-zero pairing scheme has not been discussed in detail in the literature. The pole-zero pairing problem can be seen as the availability of an additional degree of freedom for the designer to use to optimise performance. This additional degree of freedom, conceptually, affects several characteristics of concern to the designer such as sensitivity, dynamic range, total capacitance etc. Trade-offs between these characteristics must be made in any design.

It is generally a goal of the SC circuit designer to optimise circuit performance while simultaneously minimising silicon area [3–6]. Unfortunately this is such a challenging problem that most designers commit suboptimal designs to fabrication. A method of reducing and/or minimising silicon area while maintaining a given set of performance specifications for practical cascaded SC biquad structures by using an optimal pole-zero pairing strategy is discussed in this paper.

2 Development

It will be assumed that a function $H(z)$ has been obtained which meets given desired performance specifications and that a circuit is to be synthesised to realise $H(z)$. It will be further assumed that the degree of the numerator of $H(z)$ is less than or equal to that of the denominator and that $H(z)$ is to be realised as a cascade of biquadratic sections.

The conventional approach in realising $H(z)$ as a cascade of biquads is to express $H(z)$ as a product of biquadratic fractions:

$$H(z) = \prod_{k=1}^{n_1} \left( \frac{\alpha_{2k} z^2 + \alpha_{1k} z + \alpha_{0k}}{\beta_{2k} z^2 + \beta_{1k} z + \beta_{0k}} \right) = \prod_{k=1}^{n_1} H_k(z)$$

(1)

where $\alpha_{ij}$ and $\beta_{ij}$ are real for all $i, j$. The degree of $H(z)$ is $2n_1$ provided that $\beta_{2k} = 1$ for all $k$ and $2n_1 - 1$ when the $\beta_{2k}$ variables satisfy the equation

$$\beta_{2k} = \begin{cases} 1 & k < n_1 \\ 0 & k = n_1 \end{cases}$$

$H_k(z)$ represents the corresponding biquadratic fraction in $z$. The SC structure is synthesised by cascading $n_1$ biquadratic blocks with transfer functions $H_k(z)$, $k = 1, \ldots, n_1$.

It should be noted from eqn. 1 that the pole pair and zero pair pairings are not unique and that if there are real poles and/or zeros, even the pole pairings or zero pairings into biquadratic polynomials are not unique. The number of possible pole-zero pairings into biquadratic polynomials can be quite large.

To determine the number of 2nd-order pole-zero pairings, assume that all poles and zeros are unique and that there are $n_2$ poles and $n_2$ zeros, respectively. Define $\phi(k)$ and $\theta(k)$ by the expressions

$$\phi(k) = \begin{cases} \prod_{i=0}^{(k-2)/2} (k-1-2i) & \text{for } k \text{ even} \\ \prod_{i=0}^{(k-1)/2} (k-2i) & \text{for } k \text{ odd} \end{cases}$$

(2)
\[ \theta(k) = \begin{cases} \frac{k}{2} & \text{for } k \text{ even} \\ \frac{k + 1}{2} & \text{for } k \text{ odd} \end{cases} \]  

(3)

There are \( \phi(k) \) unique decompositions of a set of \( k \) elements into disjoint subsets of element pairs, and the number of disjoint subsets in each decomposition is \( \theta(k) \). Assuming the number of poles is greater than or equal to the number of zeros and that all zeros are paired (with the exception of the odd zero for \( m \) odd), it can be shown that the number of unique 2nd-order biquadratic decompositions \( H(z) \) is given by

\[ N = \phi(m) \phi(n) \frac{\theta(n)}{\theta(m)} \theta(m)! \]  

(4)

In the case that some of the real zeros are allowed to be singly decomposed rather than paired, as is often done when the number of unique 2nd-order biquadratic decompositions of \( H(z) \) increases significantly. Correspondingly, in the case that some poles or zeros are repeated, the number of unique decompositions will decrease.

To obtain an appreciation of the number of possible pole-zero pairings, it follows from eqn. 4 that if \( H(z) \) has four unique real zeros and four unique real poles, \( N = 18 \) whereas if there are six unique real zeros and six unique real poles, \( N = 1350 \).

It should be apparent that for large \( m \) and \( n \), the number of possible pairings is quite large. Many practical SC applications exist in which the number of possible pole-zero pairings is very large. The questions naturally arises, 'How should the pole-zero pairings be made?' and 'How significant are the effects of a nonoptimal pole zero pairing strategy?'

As a simple example, consider the 4th-order Chebychev bandpass approximation which has two distinct complex conjugate pole pair located at \( P_1 \) and \( P_2 \) and four real zeros. Two zeros are located at \( z = +1 \) and the other two are located at \( z = -1 \). There are three possible pole-zero pairing strategies for the biquadratic decomposition indicated pictorially in Fig. 1.

It is reasonable to anticipate that there will be appreciably different characteristics of the circuits that are synthesised based on the different decomposition schemes.

It is known that in SC filter designs there are tradeoffs between maximum voltage swing, capacitance, clock frequency and passive sensitivity [5–6]. In an attempt to isolate the pole-zero pairing effects, it will be assumed that the topology for implementing all biquadratic sections \([4, 8]\) is fixed and that the sections are all designed to have identical \( \omega_0 \) and \( Q \) sensitivities. It will also be assumed that the gain of the individual biquads is distributed by one of the following two strategies:

(a) Gain distribution strategy 1 (single frequency):

\[ |H_k(z)|_{z = e^{j\omega T}} = \left| \frac{H(z)}{\omega_0 T} \right|_{z = e^{j\omega T}}^{1/n} \]  

(5)

for some \( \omega_0 \) and for all \( 1 \leq k \leq n \). In this expression \( T \) is the clock period, \( \omega_0 \) could reasonably be chosen at the middle of the passband.

(b) Gain distribution strategy 2 (equal peak gain):

\[ \max_{\omega T} |H_k(z)|_{z = e^{j\omega T}} = H_{max} \]  

(6)

where \( H_{max} \) is the maximum gain of the \( j \)th stage.

Fig. 1 Pole-zero pairing possibilities for a 4th-order Chebychev filter

Note that both gain distribution strategies are independent of how the individual biquads are ordered in the cascade. These types of gain distribution strategies were intentionally adopted, possibly at the expense of some dynamic range reduction, to circumvent the need for considering position in the cascade as another variable. In the actual implementation, the sequence of the biquads in the cascade should be chosen to maximise dynamic range [8–10].

It is our conjecture that an optimal pole-zero pairing strategy, in the general case, is a function of:

(i) \( H(z) \)
(ii) the topology of the biquadratic block
(iii) the specific sensitivity requirements and the gain distribution strategy
(iv) the sequence of the biquadratic blocks in the cascade
(v) the system clock frequency.

An analytical treatment of the problem, in general, appears to be unwieldy. The significance of the pole-zero pairing strategy is demonstrated in the following Section by considering practical examples.

3 Significance of pole-zero pairing strategy

It will be assumed that all even-order biquadratic sections are realised with the low GB sensitivity and low power consumption structure \([4, 8]\) shown in Fig. 2 and that partial positive feedback is used in these structures to reduce total capacitance \([4–7]\). Relevant design equations appear in the Appendix. Further it will be assumed that the partial positive feedback is restricted in each block to the extent that \( |S_{oo}^{00}| \leq 1 \) and \( |S_{oo}^{00}| < 1 \) for all capacitor ratios \( x \) in the structure. The pole-zero

pairing problem is initially addressed by using AROMA [7] to synthesize $H(z)$ with the cascaded biquadratic structure using all possible pole-zero pairing schemes.

Consider initially the bandpass filter requirement depicted in Fig. 3 where $H_{\text{max}}$ is the peak gain in the passband, $x_p$ is the maximum ripple in the passband and $x_s$ is the minimum loss in the stopband. Assume for comparison purposes that $f_1 = 900 \text{ Hz}, f_2 = 1.0 \text{ kHz}, H_{\text{max}} = 0 \text{ dB}, x_p = 30 \text{ dB}$ and $x_s = 0.5 \text{ dB}$. The effects of the pole-zero pairings using a Chebyshev approximation and an elliptic approximation (with 0 dB peak gain, 0.5 dB passband ripple and 0.5 dB cutoff frequencies at $f_1$ and $f_2$) obtained from standard tables of the poles of the approximating functions [12] by using the bilinear $z$-transform of the corresponding $s$-domain approximations to obtain $H(z)$ and the two gain distribution strategies discussed previously are shown in Table 1 for several values of $n$ (the order of $H(z)$) for a clock frequency of $f_c = 8f_0 = 8\times f_1 f_2$. The stopband corners $f_3$ and $f_4$ vary with $n$. It can be seen from this Table that significant reductions in total capacitance can be made without sacrificing performance through judicious pole-zero pairings. For example, almost a 50% reduction in total capacitance was observed for the 8th-order Chebyshev single-frequency approximation by using the minimum total capacitance structure rather than the maximum total capacitance structure. Furthermore, for the Chebyshev approximation the single frequency gain distribution provides a greater reduction in total capacitance. For the elliptic approximation the opposite occurs, the greater reduction is obtained when the equal peak gain distribution is used.

From this table it can be observed that, in general, the capacitance reduction potential increases with the order of the transfer functions. It can also be seen that the total capacitance varies with the gain distribution strategy and that for a given order of approximation, the total capacitance required for the Chebyshev approximation can differ significantly from that required for the elliptic approximation. It can also be observed that for low orders the elliptic approximation requires less total capacitance, whereas for higher orders, the total capacitance required for the elliptic approximation becomes larger. Finally, although not directly related to the pole-zero pairing problem, this Table bears out the fact that the total capacitance for a given approximating function increases significantly with increasing order.

All possible pole-zero pairing strategies were exhaustively considered to obtain the data presented in Table 1 (e.g., for $n = 12, 281$ pole-zero pairings were investigated for the Chebyshev approximation). As was demonstrated previously, the number of combinations becomes unwieldy as the order of the approximation increases, making an exhaustive investigation impractical in these cases.
Others have investigated the pole-zero pairing problem in related applications [9-11]. Some investigators have been concerned with sensitivity reduction whereas others were concerned about maximizing dynamic range. In these discrete applications, the component spread, which is related to total capacitance in SC applications, did not serve as the focal point of the investigations. The component spread was not of major concern from an economic viewpoint because the actual cost of discrete components is not strongly dependent on the nominal component values in audio frequency filtering applications. These researchers were able to develop 'rules of thumb' for pole-zero pairing which resulted in reasonable filter performance. The resultant pole-zero pairing strategies were typically based on a readily computable metric relating to the relative spacing between the poles and zeros. Such a strategy is useful because the pole-zero pairings can be achieved easily with relatively minimal computational overheads.

We have been looking for a similar spacing-based pole-zero pairing strategy for total capacitance minimisation in SC circuits. To date, we have not been successful in finding such a strategy. Numerous examples involving lowpass, bandpass and highpass functions with varied approximating functions, orders, clock frequencies etc. have been investigated. Space limitations preclude discussing these investigations in this paper. Suffice it to say that in some examples pairing closely spaced poles and zeros resulted in a relatively low total capacitance whereas in other examples such a strategy resulted in a relatively large total capacitance.

3.1 Statistical approach

As an alternative, we are suggesting an exhaustive investigation when the number of pairings is small and a statistical approach when the number of combinations is large. Remarks on the statistical approach follow.

As the number of unique pole-zero pairing strategies is finite, they can be ranked ordered relative to the total capacitance $TC$ by the sequence $S_1, \ldots, S_M$, where $S_i$, $i = 1, \ldots, M$ denotes a specific pole-zero pairing strategy and $TC_i \geq TC_j$ for $i \geq j$. Relative to this ranking order, the total capacitance ratio, defined previously, is given by $TC_C = TC_M/TC_i$. Ideally, we would like to obtain the pairing strategy $S_i$.

Assume initially that $TC_C \approx (TC_M - TC_i)(M-1)/(M-i) + TC_i$. This assumption has been made because in the examples considered, a reasonably uniform distribution between $TC_1$ and $TC_M$ was observed. If it is now assumed that $K$ pole-zero pairing strategies are randomly selected from the group of $M$ possible structures, the expected value of the pairing strategy in this group of $K$ elements with minimum total capacitance is, for large $K$, approximately given by

$$TC_{C_{K, min}} \approx TC_i + \frac{(TC_M - TC_i)}{K + 1}$$  \hspace{1cm} (8)

The pole-zero pairing strategy is thus to pick $K$ pole-zero pairings randomly and select the one from this set which results in minimum total capacitance. Although the minimum total capacitance will not, in general, be obtained with this scheme, significant reductions in total capacitance relative to an arbitrary pole-zero pairing strategy can be anticipated in the average sense.

3.2 Effects of clock frequency and filter specifications on $TC_C$

It was seen in the example presented in the preceding Section that an increase of nearly 100% in total capacitance can result in practical SC applications with a non-optimal pole-zero pairing strategy. In this Section, the influence of changing clock frequencies and filter specifications will be investigated.

To investigate the effect of clock frequencies, consider for comparison purposes a 6th-order Chebyshev filter which has the same specifications as considered previously, namely $H_{max} = 0$ dB, $\alpha_p = 0.5$ dB, $\alpha_z = 30$ dB, $f_1 = 900$ Hz and $f_2 = 1000$ Hz. In Table 2, the values of $TC_C$ for this circuit with three different clock frequencies using the equal peak gain distribution strategy are presented. Note that although major differences in $TC_C$ are not observed, the pole-zero pairing does become more critical for high $f_c/f_0$ because total capacitance increases significantly with $f_c$.

Table 2: Effects of clock frequency on total capacitance for different pole-zero pairings

<table>
<thead>
<tr>
<th>$f_c$, kHz</th>
<th>$f_0$, kHz</th>
<th>$f_1$, kHz</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>16</td>
<td>32</td>
</tr>
<tr>
<td>$TC_{C_{min}}$</td>
<td>355</td>
<td>780</td>
</tr>
<tr>
<td>$TC_{C_{max}}$</td>
<td>441</td>
<td>1036</td>
</tr>
<tr>
<td>$TC_C$</td>
<td>1.24</td>
<td>1.33</td>
</tr>
</tbody>
</table>

$TC_{C_{min}}$ and $TC_{C_{max}}$ are in terms of unit capacitances.

To consider the effects of changing filter specifications, the bandwidth of the Chebyshev and elliptic bandpass responses were varied. The ripple, gain and clock frequency were held constant ($\alpha_p = 0.5$ dB, $\alpha_z = 30$ dB and $f_c = 8.4 f_0$). The $TC_C$ values are compared in Table 3. Note that although rather significant changes in the specifications were made, the $TC_C$ changes are comparable in magnitude to those observed for changing clock frequency, order, gain distribution strategy or type of approximation.

4 Conclusions

The effects of pole-zero pairing in cascaded biquad SC filters have been investigated. To compare different pole-zero pairings with the same circuit properties several design constraints were imposed. It was shown that significant reductions in total capacitance can be anticipated through an optimal pole-zero pairing scheme relative to a

Table 3: Effects of changing filter specifications on $TC_C$

<table>
<thead>
<tr>
<th>$BW$, Hz,</th>
<th>Chebyshev</th>
<th>Elliptic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_c$, Hz</td>
<td>900</td>
<td>900</td>
</tr>
<tr>
<td>$f_0$, Hz</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>$f_{max}$</td>
<td>1024</td>
<td>1024</td>
</tr>
<tr>
<td>$f_{min}$</td>
<td>1024</td>
<td>1024</td>
</tr>
<tr>
<td>$TC_{C_{max}}$</td>
<td>441</td>
<td>441</td>
</tr>
<tr>
<td>$TC_{C_{min}}$</td>
<td>1036</td>
<td>1036</td>
</tr>
<tr>
<td>$TC_C$</td>
<td>1.24</td>
<td>1.24</td>
</tr>
</tbody>
</table>

$TC_{C_{max}}$ and $TC_{C_{min}}$ are in terms of unit capacitances.
random pairing strategy without deterioration in filter performance. The reduction in total capacitance should result directly in a silicon area reduction at fabrication. The importance of considering the pole-zero pairing problem becomes more significant both at high clock frequencies and as the degree of the approximating function \( H(z) \) increases. From the examples presented, it appears that the magnitude of the pole-zero pairing problem is not strongly dependent on the exact filter specifications or gain distribution strategy.

For higher-order \( H(z) \) the number of possible pole-zero pairing combinations becomes unwieldy. Although an exhaustive computer search may be justified in some cases, a simple statistical approach which offers, on average, appreciable reductions in total capacitance was introduced.

6 References


6 Appendix: Design equations for the SC biquad

For the transfer function when input and output are sampled at the same time for the SC biquad of Fig. 2 is given by

\[
H(z) = \frac{z^2 - z_1 + z_2 + z_3 + z_4}{1 + z_8}
\]

Eqn. 9 can be also written, for convenience, as

\[
H(z) = K \frac{z^2 + ez + \delta}{z^2 - 2r \cos \theta z + r^2}
\]

A simple solution of the design equations for the poles follows. Let \( a_9 = ka_8 \) and \( 0 \leq k < 1 \):

\[
a_9 = \frac{1 - r^2}{r^2/k - 1}
\]

select \( a_2 \) and obtain

\[
a_7 = \frac{1}{1 + a_8}(1 + r^2 - 2r \cos \theta a_2)
\]

\( k \) is related to \( S_{21} \), i.e., \( k = S_{21}^0/(1 + S_{12}^0) \). Table 4 shows the zero placement formulas for different type of filters used in this paper. The notation used is described on p. 163 of Reference 2. \( K = a_g/(1 + a_g) \) unless otherwise specified.

### Table 4: Equations for filters used in the paper

<table>
<thead>
<tr>
<th>Filter type</th>
<th>Design equations</th>
<th>Simple solution</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>LP 20</strong></td>
<td>( a_g - a_5 a_2 a_5 )</td>
<td>( a_5 = 0, a_6 = a_5 )</td>
</tr>
<tr>
<td></td>
<td>( a_5 / a_6 )</td>
<td>( a_9 = K(1 + a_8) )</td>
</tr>
<tr>
<td></td>
<td>( a_5 + a_6 - a_2(a_1 + a_6) )</td>
<td>( a_1 a_2 = 4a_6 )</td>
</tr>
<tr>
<td><strong>BP 20</strong></td>
<td>( a_5 - a_5 a_2(a_1 + a_6) )</td>
<td>( a_5 = 0, a_6 = a_5 )</td>
</tr>
<tr>
<td></td>
<td>( a_5 / a_6 )</td>
<td>( a_6 = K(1 + a_8) )</td>
</tr>
<tr>
<td><strong>HP</strong></td>
<td>( a_5 - a_5 a_6 )</td>
<td>( a_6 = 0, a_5 = a_6 )</td>
</tr>
<tr>
<td></td>
<td>( a_5 / a_6 )</td>
<td>( a_5 = K(1 + a_8) )</td>
</tr>
<tr>
<td><strong>General</strong></td>
<td>( K = a_g / (1 + a_8) )</td>
<td>( a_9 = K(1 + a_8) )</td>
</tr>
<tr>
<td>(except BP 20)</td>
<td>( a_5 = 0, a_6 = a_5 )</td>
<td>( a_5 = K(1 + a_8) )</td>
</tr>
<tr>
<td></td>
<td>( a_5 + a_6 = a_5 a_6 )</td>
<td>( a_5 = K(1 + a_8) )</td>
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<td></td>
<td>( a_5 / a_6 )</td>
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<td></td>
<td>( a_5 + a_6 = a_5 a_6 )</td>
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<tr>
<td></td>
<td>( a_5 / a_6 )</td>
<td>( a_5 = K(1 + a_8) )</td>
</tr>
</tbody>
</table>


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