Frequency Response Measurement Algorithms

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Abstract

Two frequency response measurement algorithms are reviewed. One is based on the FFT algorithm, and the other uses least squares techniques based on a first-order moving average model. A least squares method based on a first-order autoregressive model is also presented in this paper, and these three methods are compared on the basis of their implementation cost and accuracy. Extensive simulations based on the Monte Carlo techniques have demonstrated that all three methods show similiar performance. Thus, the methods based on least squares techniques which require low implementation cost for data acquisition will be able to serve as low-cost and high-accuracy frequency response measurement techniques.

1 Introduction

Frequency response measurements at discrete frequencies are required for frequency-domain system identifications [1]. System identifications are necessary for many applications. For example, a frequency-domain system identification plays an important role in tuning of continuous-time filters [2],[3]. The accuracy of tuning highly depends on the accuracy of the system identification, which means that accurate frequency response measurements are crucial for continuous-time filter tuning. The implementation cost as well as the accuracy of the measurement methods should be considered because the cost and the accuracy are usually in a trade-off relation.

Two frequency response measurement algorithms have been reported in literature. They use sinusoidal inputs and collect input and output time-domain samples for further interpretation. One of them uses FFT algorithms [4], and the other uses least squares algorithms [5] based on a first-order moving average (MA) model to estimate the frequency responses. The FFT method requires a large number of consecutive samples and thus, requires high-speed A/D converters and sample-and-hold circuits. In contrast, in the least squares method each data set contains only two consecutive input samples and an output sample and can be grabbed randomly or asynchronousely. Thus, the least squares method requires lower cost for data acquisition hardware implementation than the FFT method.

In this paper a least squares (LS) algorithm based on a first-order auto-regressive moving average (ARMA) model is presented and compared with the two algorithmes mentioned above by extensive simulations based on Monte Carlo methods.

2 Discussion of Mehtods

In this section the three frequency response measurement algorithms are presented and discussed. All three methods use time-domain input and output samples to measure the frequency responses. If a linear continuous-time system is excited by a sinusoidal input

$$x(t) = Acos(\omega_o t) \tag{1}$$

then the output, in the steady state, can be described by

$$y(t) = GAcos(\omega_o t + \phi) \tag{2}$$

where G and ϕ are the gain and phase responses of the system at frequency ω_o. The general frequency response measurement problem is to find G and ϕ at a finite number of frequencies. Since experimentally obtained data are inevitably contaminated by some noise and system nonlinear effects, a reasonable approach is to find G and ϕ which best fit the given data.

2.1 FFT Method

In the FFT method [4] the input and output are recorded for N_f sample sets at sampling frequency $f_s = 1/T_s$ from which the frequency response at one point is calculated. Each set contains two samples. N_f must be a power of 2. The output estimate can be defined as

$$\begin{aligned} \hat{y}(kT_s) &= GAcos(\omega_o kT_s + \phi) \\ &= G_c cos(\omega_o kT_s) + G_s sin(\omega_o kT_s) \end{aligned} \tag{3}$$

where

$$G_c = GAcos(\phi) \tag{4}$$

$$G_s = GAsin(\phi) \tag{5}$$

It can be easily seen from (4) and (5) that if G_c ,G_s and A are found, then the gain and phase responses can be calculated by

$$G = \frac{1}{A}\sqrt{G_c^2 + G_s^2} \tag{6}$$

$$\phi = -tan^{-1}\left(\frac{G_s}{G_c}\right) \tag{7}$$

The estimated G_c and G_s which best fit to the data in the least squares sense are closely related to the DFT/FFT of $y(kT_s)$ if the test frequencies are selected to be $\omega_l = 2\pi l/N_f T_s$ for integer l. The DFT/FFT of $y(kT_s)$ is

$$\begin{aligned} Y_n = FFT(y) &= \sum_{k=0}^{N_f-1} GAcos\left(\frac{2\pi lk}{N_f} + \phi\right) e^{-j(2\pi nk)/N_f} \\ &= \begin{cases} \frac{N_f}{2}(G_c - jG_s), & n = l \\ 0, & n \neq l \end{cases} \end{aligned} \tag{8}$$

The DFT/FFT of the input $x(kT_s)$ is given by

$$\begin{aligned} X_n = FFT(x) &= \sum_{k=0}^{N_f-1} Acos\left(\frac{2\pi lk}{N_f}\right) e^{-j(2\pi nk)/N_f} \\ &= \begin{cases} \frac{N_f A}{2}, & n = l \\ 0, & n \neq l \end{cases} \end{aligned} \tag{9}$$

For $n = l$, the ratio of Y_l to X_l is given by

$$\begin{aligned} \frac{Y_l}{X_l} &= \frac{G_c - jG_s}{A} \\ &= Ge^{j\phi} \end{aligned} \tag{10}$$

Thus, in the FFT method the gain and phase responses at $\omega_l = 2\pi l/N_f T_s$ are obtained by the ratio of the FFT of $y(kT_s)$

to the FFT of $x(kT_s)$ for $n = l$.

The FFT method estimates the frequency responses accurately in a fast way by the effective FFT algorithm, but using the FFT on $y(kT_s)$ and $x(kT_s)$ for only one frequency point is somewhat inefficient. This inefficiency can be alleviated by using a sinusoidal input of which the frequency changes from a starting value to a final value so as to estimate the frequency responses at several frequencies at once. However, the accuracy might be degraded at fixed N_f compared to that of the original method. Since the FFT method uses a large number of consecutive data, a fast data acquisition system is required for high frequency measurements to avoid the aliasing problem.

2.2 LS Method with a MA Model

In this algorithm proposed by Loh [5], a first-order MA model is used to determine the gain and phase responses by the LS algorithm. If the input $x(t)$ is sampled at $t = t_o$ and $t = t_o - T_s$, and the output $y(t)$ at $t = t_o$, then the relationship between the output sample and the input samples can be given by

$$y(t_o) = b_0 x(t_o) + b_1 x(t_o - T_s) \tag{11}$$

where

$$b_0 = G[cos\phi + sin\phi cot(\omega_o T_s)] \tag{12}$$

$$b_1 = -G\frac{sin\phi}{sin(\omega_o T_s)} \tag{13}$$

Equation (11) is valid for all t_o provided $\omega_o T_s \neq n\pi$ for integer n. By taking the Fourier Transformation on (11) the frequency response at ω_o is given by

$$Ge^{j\phi} = b_0 + b_1 e^{-j\omega_o T_s} \tag{14}$$

If there is no measurement error, only two data sets are enough to determine b_0 and b_1 and thus, the frequency response exactly. Taking into account the measurement errors, a large number of data sets can be used to perform the well known LS algorithm. This gives an estimation of b_0 and b_1 which best fit to the given data. The estimated coefficient vector can be then described by a LS solution

$$\hat{\mathbf{c}}_m = [\mathbf{A}^T\mathbf{A}]^{-1}[\mathbf{A}^T\mathbf{y}_0] \tag{15}$$

where $\hat{\mathbf{c}}_m = [b_0, b_1]^T$ and $\mathbf{A} = [\mathbf{x}_0, \mathbf{x}_{-1}]$, and $\mathbf{x}_{-1}$ is a sampled input vector, and $\mathbf{x}_0$ and $\mathbf{y}_0$ are the input and output data vectors sampled with a delay T_s. Vectors, $\mathbf{x}_0$, $\mathbf{x}_{-1}$ and $\mathbf{y}_0$ have dimention N_m, where N_m is the number of data sets, and each data set contains 3 samples.

In this method each data set contains only two consecutive input samples and one output sample. Since the data sets can be collected randomly or asynchronously, the data acquisition system can be implemented with three fast sample-and-hold circuits (two for input and one for output) and a slow A/D converter. It has been demonstrated by simulations and experiments that this technique can accurately measure the frequency responses even with noisy and low-resolution data [5].

2.3 LS Method with an ARMA Model

It can be seen in the previous subsections that the LS method requires lower cost for data acquisition than the FFT method. In this subsection a first-order AutoRegressive Moving Average (ARMA) model instead of a first-order MA model is applied for the LS frequency response measurement. To utilize a first-order ARMA model one more output sample must be added to the data set of the MA model. Each data set of the ARMA model thus includes two input samples and two output samples.

The input $x(t)$ and ouput $y(t)$ are sampled at $t = t_o$ and $t = t_o - T_s$ by four sample-and-hold circuits, and these four samples constitute one data set. Once all four samples are converted by an A/D converter which does not have to be fast, another data set is sampled with time delay T_{int}. Of course, the time interval, T_{int}, must be selected to be long enough for the A/D converter to finish conversion of four sampled and held data. T_{int} does not have to be the same for the whole data acquisition period. The relationship between output samples and input samples for any one data set can be readily obtained as follows by applying basic trigonometric identities to (1) at $t = t_o - T_s$:

$$y(t_o) = a_1 y(t_o - T_s) + b_0 x(t_o) + b_1 x(t_o - T_s) \tag{16}$$

where

$$a_1 = 1/cos(\omega_o T_s) \tag{17}$$

$$b_0 = Gcos(\omega_o T_s + \phi)/cos(\omega_o T_s) \tag{18}$$

$$b_1 = -Gcos\phi/cos(\omega_o T_s) \tag{19}$$

Equation (16) is valid for all t_o provided $\omega_o T_s \neq (2n+1)\pi/2$ for integer n. By taking the Fourier Transformation on (16) the frequency response at ω_o is given by

$$Ge^{j\phi} = \frac{b_0 + b_1 e^{-j\omega_o T_s}}{1 + a_1 e^{-j\omega_o T_s}} \tag{20}$$

With ideal measurement and an ideal system, only three data sets are required to determine the three unknowns, b_1, a_0 and a_1. With nonideality consideration, ample data sets are fed to the LS algorithm to estimate the coefficients. The LS solution can be obtained in a similiar way as before.

$$\hat{\mathbf{c}}_a = [\mathbf{A}^T\mathbf{A}]^{-1}[\mathbf{A}^T\mathbf{y}_0] \tag{21}$$

where $\hat{\mathbf{c}}_a = [a_1, b_0, b_1]^T$ and $\mathbf{A} = [\mathbf{y}_{-1}, \mathbf{x}_0, \mathbf{x}_{-1}]$, and $\mathbf{x}_{-1}$ and $\mathbf{y}_{-1}$ are sampled input and output data vectors, and $\mathbf{x}_0$ and $\mathbf{y}_0$ are the input and output data vectors sampled with a delay T_s. The vectors have dimention N_a, where N_a is the number of data sets, and each data set contains 4 samples. In this method the data acquisition system can be implemented with one more sample-and-hold circuit added to the system for the MA model.

If the signals are sampled at the same rate for all data sets, i.e., T_{int}=T_s, as in the FFT method, then we can increase the number of data sets for the fixed total sample number. One data set can be obtained at every sample time since one input and output sample set contributes to two data sets as previous data and as current data. This is the case as the general time-domain LS system identification methods do, where for the nth-order ARMA model $N - n + 1$ data sets can be obtained with N input and output samples. For our first-order case $(2N_a)$ data sets can thus be used for the LS problem with increased dimention $2N_a$ of the vectors in (21). This will lead to improved accuracy beacuse the accuracy of the linear LS problem is usually proportional to the number of data, but more cost for data acquisition will be required to grab a large number of consecutive data.

Since the cost and the accuracy are in a trade-off relation, one of the two strategies can be selected according to which has a higher priority. This may be one advantage of the ARMA method over the MA method because in the MA method the number of data sets can not be increased through sampling with $T_{int} = T_s$, or even though it can be increased by adding one more sampe-and-holder, one output sample can contribute to only one data set.

3 Simulation Results

In this section the three frequency response measurement algorithms are compared through extensive Monte Carlo based simulations. The nonidealities of the system under test, the data acquisition system, and the excitation signals are included in this simulation.

The measurement noise associated with the data acquisition process such as quantization noise and system noise are modelled as uniformly distributed additive random numbers. If the measurement noise is distributed as $U(-\epsilon_n, \epsilon_n)$, and the signal amplitude is A, then the variances of the signal and the noise are

$$\sigma_s^2 = \frac{A^2}{2} \qquad \sigma_n^2 = \frac{\epsilon_n^2}{3} \tag{22}$$

The signal to noise ratio (SNR) is then defined as

$$SNR = 10log_{10}\left(\frac{\sigma_s^2}{\sigma_n^2}\right) \tag{23}$$

$$= 10[log_{10}(1.5) - 2log_{10}(A/\epsilon_n)] \tag{24}$$

The SNR will be about 40dB for $\epsilon_n = 0.01$ and $A = 1$. The excitation signal nonlinearity and the system nonlinearity are approximated by the second harmonic distortion, and the higher-order distortions are neglected. The total harmonic distortion (THD) of the excitation input signal is always set to -40dB. The default value of the SNR is 40dB, and the default THD of the output signal due to the system nonlinearity is -40dB.

First, the frequency response of a second-order bandpass filter is measured. The system is excited by a sinusoidal input at its resonant frequency. The ideal gain and phase responses of the system at the frequency are $G = 1$ and $\phi = 0°$. The amplitude of the input is set to unity. The simulation results for the bandpass filter are shown in Fig.1, 2 and 3. All statistical results are obtained from 100 independent trials, and the y axes of all figures have a log scale.

Fig.1 shows the standard deviations of the gain and phase errors when the SNR and the input signal and system THD's are set to their default values. The accuracies of the three methods are all improved as the number of data sets is increased. Around 100 data sets seem to be an optimum point for all three cases since the accuracy improvement are very slow with further increase of data sets. The MA algorithm shows nonmonotonocity in the phase error plot shown in Fig.1(b). Note that the data sets have different number of samples, i.e., each data set of the FFT, MA, and ARMA methods has 2, 3, and 4 samples, respectively.

Fig.2 shows the standard deviations at various SNR values. In this case the same total number of samples are used to compare the algorithms more fairly. The total sample number is 128. The gain and phase measurement errors are monotonously decreased as the SNR is increased. All three algorithms show similiar accuracies and can tolerate fairly noisy environment. The gain error less than 1% and the phase error less than 0.5° can be obtained up to as low as 30dB SNR. The MA algorithm shows slightly better accuracy at more noisy environment than the ARMA algorithm.

In Fig.3 the effects of the system nonlinearity on the measurement are shown. In this case the SNR is set to 60dB in order to investigate the impact of the harmonic distortion more clearly. The THD of the excitation signal is $-40dB$. It can be observed from Fig.3 that the gain error of the FFT algorithm and the phase error of the MA algorithm are more sensitive to the system nonlinearity compared to others. Both gain and phase errors of the ARMA algorithm are shown to be very insensitivity to the system nonlinearity.

Up to now the three algorithms are compared at one frequency point where the ideal gain is 1, and the ideal phase is 0 degree. Now, a simple second-order lowpass filter of which the transfer function is

$$H(s) = \frac{1}{s^2 + s + 1} \tag{25}$$

is tested at the passband through the stopband, where the normalized cutoff frequency is at 1 (rad/sec). Fig.4 shows the standard deviations of the gain errors with respect to the maximum gain of the system and the phase measurement errors in degree. The nonideal parameters are set to their default values. The MA method shows better results than any others through the almost entire frequency range.

4 Conclusions

Two frequency response measurement algorithms, the FFT method and the LS method based on a first-order MA model, have been reviewed and discussed. A LS method based on a first-order ARMA model has been presented and comparatively discussed with the other methods from the cost point of view and from the accuracy point of view. The LS methods require lower cost for implementation of data acqusition systems than the FFT method because they do not require a large number of consecutive samples, but require a number of data sets which consist of only two consecutive input and output samples.

Extensive simulations based on Monte Carlo techniques have demonstrated that the three algorithms show similiar performance. The ARMA method is very insensitive to the system nonlinearity, while the MA method can tolerate more noisy environment. It has been shown that the LS methods have advantages over the FFT method in the accuracy as well as the implementation cost. For a simple test lowpass filter the LS methods have shown accurate measurement results at reasonable measurement environment. For 40dB SNR and $-40dB$ THD, the gain error less than 1% and the phase error less than 1° were obtained on the normalized frequency range from dc to 2 (rad/sec). Thus, these LS methods can be well applied for the frequency-domain system identification algorithm [1],[3], where very good results can be achieved for 1% measurement errors.

References

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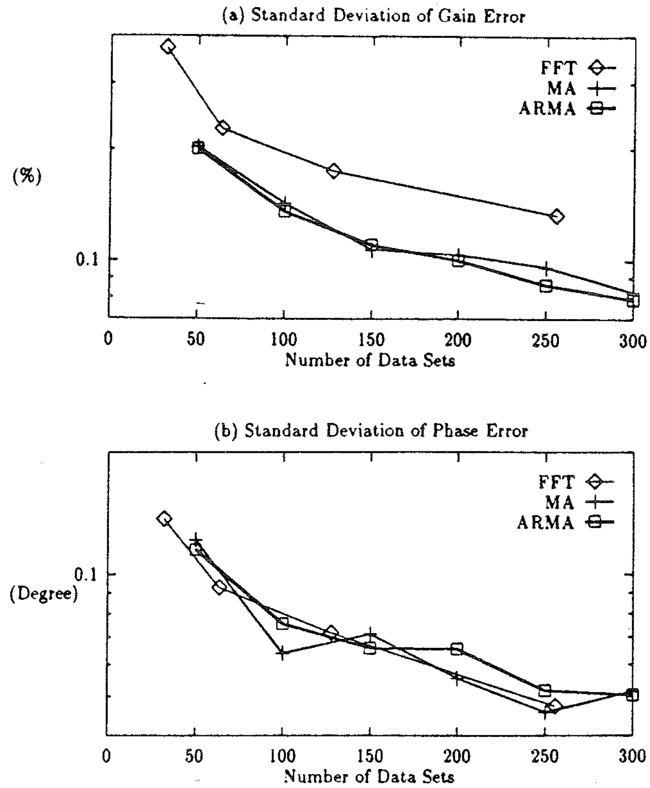

Fig.1: Simulated frequency response measurement errors versus the number of data sets (a) Gain errors (b) Phase errors

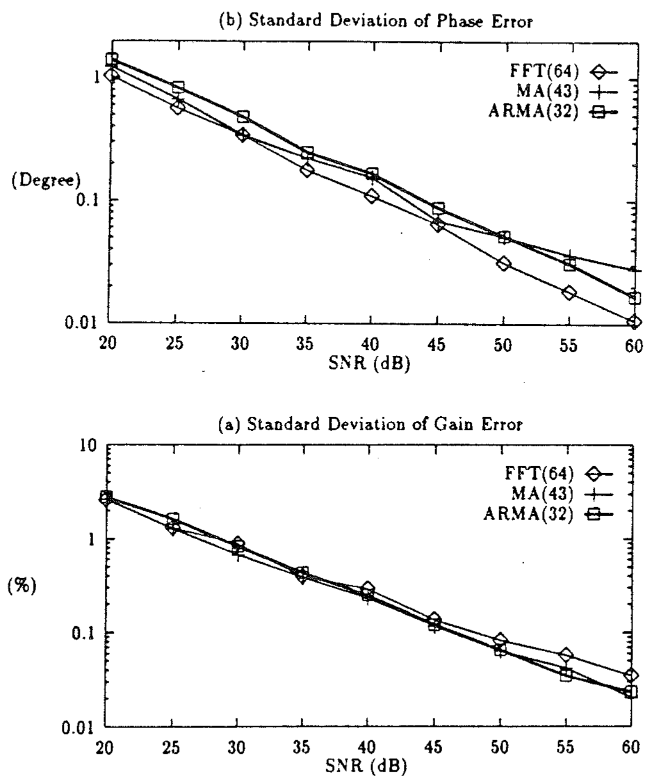

Fig.2: Simulated frequency response measurement errors versus the signal to noise ration (SNR) (a) Gain errors (b) Phase error

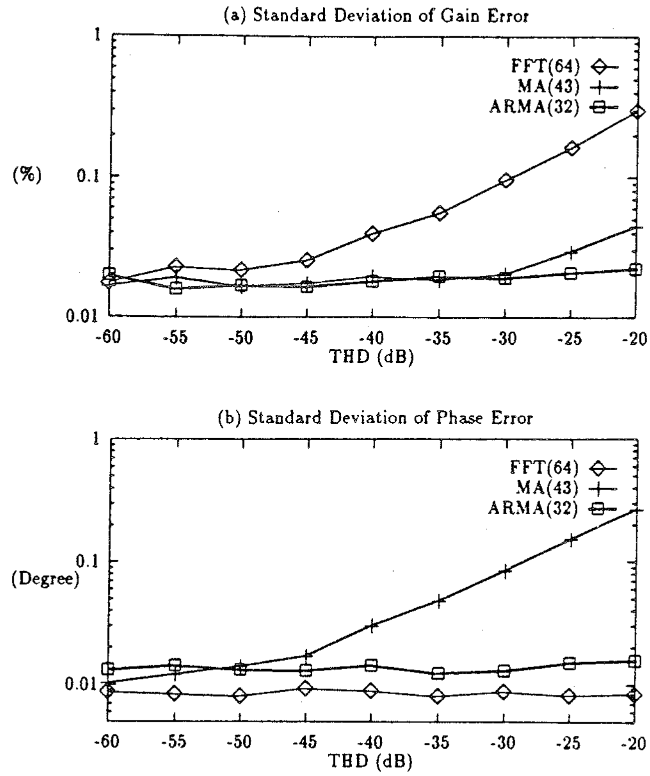

Fig.3: Simulated frequency response measurement errors versus the system nonlinearity (THD) (a) Gain errors (b) Phase errors

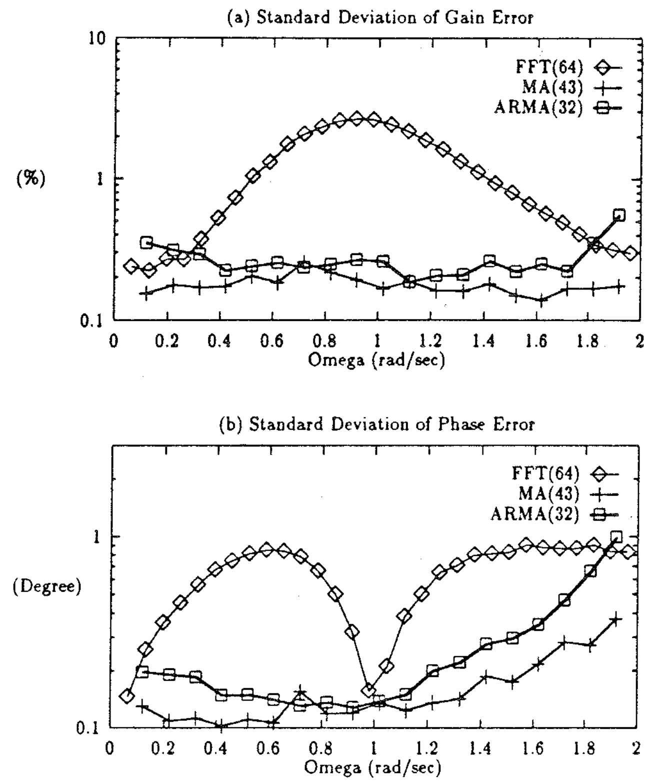

Fig.4: Simulated frequency response measurement errors for a simple lowpass filter versus frequency (a) Gain errors (b) Phase errors