A Matrix Kharitonov Theorem and Robust Control of Mechanical Systems

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ABSTRACT

This paper studies robust control of multi-body mechanical systems. A robust stability condition is presented together with a simple procedure for synthesizing such robust PID controllers. The Kharitonov theorem is generalized to polynomials with matrix coefficients bounded between two symmetric positive definite matrices of the same shape. Its restriction to the third order case applies to multi-variable PID-controlled robotic manipulators and it states roughly that a controller designed for an upper bounding inertia matrix results in stable set-point regulation for all other inertias. Furthermore, an example is presented to show that our result cannot be generalized to polynomials with positive symmetric matrix coefficients lying in general matrix intervals.

1. Problem Formulation

Consider the dynamics of a multi-degree-of-freedom mechanical system (e.g. robot manipulator) given by

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + G(\theta) = F \qquad (1)$$

where θ is a vector of joint displacements, $M(\theta)$ is the configuration-dependent symmetric positive definite inertia matrix, $C(\theta, \dot{\theta})$ are centrifugal and coriolis forces (quadratic in $\dot{\theta}$), $G(\theta)$ is the gravity force vector and F is the vector of applied joint forces. If a multivariable PID set-point controller with gravity compensation is used, the control is given by

$$F = -K_I \int e(\tau)d\tau - K_P e - K_D \dot{e} + G(\theta) \qquad (2)$$

where $e = \theta - \theta_d$ and θ_d is the *constant* set-point and K_D, K_P, K_I are symmetric (usually diagonal in practice) controller gain matrices. Defining $\varepsilon \triangleq \int e(\tau)d\tau$ and linearizing the combined dynamics of the robot and controller about the equilibrium point $(\varepsilon, \theta, \dot{\theta}) = (0, \theta_d, 0)$ yields

$$M(\theta_d)\, \varepsilon^{(3)} + K_D \ddot{\varepsilon} + K_P \dot{\varepsilon} + K_I \varepsilon = 0. \qquad (3)$$

If these linearized dynamics are exponentially stable, then Lyapunov's indirect method (see [1] page 179) implies the (local) exponential stability of the equilibrium point $(0, \theta_d, 0)$ for the combined nonlinear system (1) and (2). However, with θ_d time-varying, stability is not guaranteed. The interested reader is referred to [2] and [3] for approaches to the nonlinear tracking control problem.

The problem addressed in this paper is that of finding set-point controller gains K_P, K_I, K_D such that (3) is stable for each *fixed* θ_d. Our results only provide theory for the set-point regulation problem. However, the results can provide guidance to practitioners who iteratively tune PID tracking controller gains at a family of set-points representing the range of robot inertias.

We begin by designing controller gains which stabilize the system (3) for each fixed θ_d. Since robots and other mechanical systems usually have revolute joints, or prismatic joints with limited motion, the set of inertias is assumed to be continuously parameterized by a parameter in a compact set, Θ. Thus, there exist positive definite symmetric matrices $\bar{M}$ and $\underline{M}$ such that $\underline{M} \le M(\theta) \le \bar{M}$ for all $\theta \in \Theta$. This raises two important Kharitonov-like stability questions. (1) Under what conditions will the PID regulated robot manipulator be stable for the whole class of inertias? (2) How do we design a robust stabilizing controller if bounds on the inertia matrix are known?

Question (1) is answered in section 2 where a simple sufficient condition for robust stability of a large class of PID regulated mechanical systems is derived. Question (2) is answered in section 3 by a new procedure for designing a stabilizing controller. In addition it is shown that a controller designed based on an inertia matrix larger than all others in the family stabilizes the entire class. This is generalized to a new matrix Kharitonov theorem with all bounding matrices having the same shape. A two-link planar manipulator example is given in section 4 to illustrate the results. In section 5, we show via a counterexample that our result cannot be further extended to other types of bounding matrices. Our conclusions are made in section 6.

2. Application of Kharitonov's Theorem

Kharitonov's theorem [4] provides a powerful criterion for the strict Hurwitz property of a family of polynomials with coefficients lying within given intervals. This well-known theorem states that the strict Hurwitz property of the entire family is equivalent to the strict Hurwitz property of four specially constructed vertex polynomials. This number can be reduced for polynomials of degree less than six [5, 6].

Here we apply the simplification of Kharitonov's theorem for third order polynomials to find a robust stability condition for PID controlled robot manipulators. It is of interest to ascertain whether or not the stability of the family of polynomials can be determined by checking only some extremal polynomials. The specialization of Kharitonov's theorem to third order interval polynomials tests the stability of the whole family with just one vertex polynomial [6].

We now extend these ideas to our mechanical system (3). The characteristic equation of (3) is easily computed to be

$$\chi(s) = det[M(\theta_d)s^3 + K_D s^2 + K_P s + K_I]. \qquad (4)$$

Let λ be a root of $\chi(s)$ for the fixed $\theta_d \in \Theta$. Then there exists an associated "mode shape" v with unit 2-norm satisfying

$$[M(\theta_d)\lambda^3 + K_D\lambda^2 + K_P\lambda]v = 0. \quad (5)$$

Multiplying this equation on the left by the conjugate transpose of v, v^*, yields a polynomial in λ with real coefficients

$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \quad (6)$$

where

$$a_3 = v^* M(\theta_d)v, a_2 = v^* K_D v,\ a_1 = v^* K_P v, a_0 = v^* K_I v. \quad (7)$$

Observe that

$$\begin{aligned} a_3 &\in [\lambda_{\min}(\underline{M}), \lambda_{\max}(\bar{M})] \triangleq [\underline{a}_3, \bar{a}_3] \\ a_2 &\in [\lambda_{\min}(K_D), \lambda_{\max}(K_D)] \triangleq [\underline{a}_2, \bar{a}_2] \\ a_1 &\in [\lambda_{\min}(K_P), \lambda_{\max}(K_P)] \triangleq [\underline{a}_1, \bar{a}_1] \\ a_0 &\in [\lambda_{\min}(K_I), \lambda_{\max}(K_I)] \triangleq [\underline{a}_0, \bar{a}_0]. \end{aligned} \quad (8)$$

Equation (6) is therefore an interval polynomial. The stability of this interval polynomials can be verified by checking just *one* of the Kharitonov polynomials. According to [6], equation (6) is Hurwitz if the following polynomial is Hurwitz.

$$\chi(\lambda) = \bar{a}_3\lambda^3 + \underline{a}_2\lambda^2 + \underline{a}_1\lambda + \bar{a}_0. \quad (9)$$

The test can be further simplified by using the Routh-Hurwitz stability test which requires

$$\begin{aligned} &(i)\ \bar{a}_0 > 0,\ \underline{a}_2 > 0,\ \bar{a}_3 > 0 \\ &(ii)\ \underline{a}_1\underline{a}_2 - \bar{a}_0\bar{a}_3 > 0. \end{aligned} \quad (10)$$

Conditions (i) is satisfied if K_I , K_D , and $\bar{M}$ are positive definite. Therefore, under the assumption of symmetric positive definite gains, the closed-loop system given by equation (4) is stable if

$$\lambda_{\max}(\bar{M})\, \lambda_{\max}(K_I) < \lambda_{\min}(K_D)\, \lambda_{\min}(K_P). \quad (11)$$

The assumption of symmetric positive definite gains can be justified by the design procedure described in the next section. Note that this is only a sufficient condition, but this condition is tight for some practical numerical experiments. Nevertheless, conservatism in the condition of (11) can be reduced by using a scaling technique before applying the test, that is, (4) is stable if

$$\lambda_{\min}(QK_DQ')\lambda_{\min}(QK_PQ') > \lambda_{\max}(Q\bar{M}Q')\lambda_{\max}(QK_IQ') \quad (12)$$

for some nonsingular Q. This is true because pre- and post-multiplying the matrices in equation (4) by a non-singular matrix does not affect the stability of the characteristic polynomial. For example, with

$$M = K_D = K_P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\ K_I = \begin{bmatrix} 1 & 0 \\ 0 & 0.9 \end{bmatrix},$$

(4) is clearly stable, but (11) fails. However, with $Q = M^{-1/2}$, (12) is satisfied. In practice, if (11) fails, a good initial choice of Q is to simultaneously diagonalize two of the four matrices, making one the identity.

3. Controller Synthesis

In this section we describe a simple procedure of choosing PID gains and show that a controller design based on an upper bounding inertia matrix stabilizes the entire class of inertias. Since the roots of the characteristic equation (4) (with M replaced by $\bar{M}$) remain unchanged if the matrix polynomial is pre- and post-multiplied by $\bar{M}^{(-1/2)} \triangleq Q$, we have

$$\chi(s) = \det[Is^3 + K_D' s^2 + K_P' s + K_I'] \quad (13)$$

where

$$K_D' = QK_DQ,\ K_P' = QK_PQ,\ K_I' = QK_IQ.$$

Since the transformed inertia matrix is the identity matrix, the design can be treated as a set of n decoupled PID controller designs which can be accomplished by, say, pole placement. We choose the primed gain matrices to be scalar multiples of the identity matrix:

$$K_D' = k_d I \quad , \quad K_P' = k_p I \quad , \quad K_I' = k_i I \quad (14)$$

The controller gains used for implementation are therefore

$$K_D = k_d\bar{M}, \quad K_P = k_p\bar{M}, \quad K_I = k_p\bar{M} \quad (15)$$

A surprising fact is summarized in the following theorem.

Theorem 1: Let $\chi_0(s) = det[\bar{M}s^3 + K_D s^2 + K_P s + K_I]$ be the characteristic equation of a PID-controlled mechanical system with the controller gain matrices chosen based on the above procedure. Then the characteristic equation

$$\chi(s) = det[Ms^3 + K_D s^2 + K_P s + K_I]$$

is Hurwitz for all M: $\bar{M} \geq M > 0$, if and only if $\chi_0(s)$ is Hurwitz.

Proof: The "only if" part is trivial and the "if" part is done as follows. Since $det[\bar{M}s^3 + k_d\bar{M}s^2 + k_p\bar{M}s + k_i\bar{M}] = det[(s^3 + k_d s^2 + k_p s + k_i)\bar{M}]$ is Hurwitz by design, the Routh-Hurwitz stability test yields the stability condition

$$k_i < k_d k_p. \quad (16)$$

Let λ be a root of the characteristic equation

$$det[Ms^3 + k_d\bar{M}s^2 + k_p\bar{M}s + k_i\bar{M}] = 0.$$

Then there exists a vector v of unit 2-norm satisfying

$$[M\lambda^3 + k_d\bar{M}\lambda^2 + k_p\bar{M}\lambda + k_i\bar{M}]v = 0 \quad (17)$$

$$\Leftrightarrow v^*Mv\lambda^3 + k_d v^*\bar{M}v\lambda^2 + k_p v^*\bar{M}v\lambda + k_i v^*\bar{M}v = 0 \quad (18)$$

$$\Leftrightarrow \frac{v^*Mv}{v^*\bar{M}v}\lambda^3 + k_d\lambda^2 + k_p\lambda + k_i = 0. \quad (19)$$

Since $M < \bar{M}$, we have $v^*Mv < v^*\bar{M}v$. This together with equation (16) gives

$$\frac{v^*Mv}{v^*\bar{M}v} k_i < k_d k_p \quad (20)$$

which implies the stability of $\chi(s)$ and the proof is complete.□

For convenience, let us say that a Hurwitz polynomial has *stability degree* σ if $\max(\mathrm{Re}(\lambda_i)) \leq -\sigma$ where the λ_i's are the characteristic values. Then we have the following.

Corollary 2: If $\chi_0(s)$ has stability degree σ and $k_p \geq 2\sigma k_d$, then $\chi(s)$ also has stability degree σ for all $M \leq \bar{M}$, where $\chi(s)$ and $\chi_0(s)$ are as defined in the theorem.

Proof: This follows directly from a change of coordinates $s \to s - \sigma$ and some straight forward algebra.□

Now suppose we have unmodeled structural damping and structural stiffness in equation (1). Will our system still be stable for all configurations? The answer is positive.

Corollary 3: Let $K'_{D_s} = K_{D_s} \geq 0$ be a structural damping matrix and $K'_{P_s} = K_{P_s} \geq 0$ be a structural stiffness matrix.

$$\chi(s) = det[Ms^3 + (K_D + K_{Ds})s^2 + (K_P + K_{Ps})s + K_I]$$

is Hurwitz for all $M \leq \bar{M}$, $K_{D_s} \geq 0$, $K_{P_s} \geq 0$, if and only if

$$\chi_0(s) = det[\bar{M}s^3 + K_D s^2 + K_P s + K_I]$$

is Hurwitz where $K_D = k_d \bar{M}$, $K_P = k_p \bar{M}$, and $K_I = k_i \bar{M}$.

Proof: The proof follows exactly the same lines as the proof of the theorem, all coefficient matrices are assumed symmetric positive definite.□

In fact, Theorem 1 can be generalized to the following matrix Kharitonov theorem.

Theorem 4: Let

$$\chi(s) \triangleq det[A_n s^n + A_{n-1}s^{n-1} + \cdots + A_1 s + A_0],$$

where the coefficient matrices are all symmetric positive definite and bounded by symmetric positive definite matrices as follow: $\bar{A}_i \geq A_i \geq \underline{A}_i > 0$ for all i. Furthermore, all the bounding matrices have the same "shape", meaning that $\bar{A}_i = \bar{a}_i A$, $\underline{A}_i = \underline{a}_i A$ for all i and some symmetric positive definite matrix A. Then $\chi(s)$ is Hurwitz if and only if the following four polynomials are Hurwitz:

$$\chi_1(s) \triangleq det[\bar{A}_n s^n + \bar{A}_{n-1}s^{n-1} + \underline{A}_{n-2}s^{n-2} + \underline{A}_{n-3}s^{n-3} + \cdots],$$

$$\chi_2(s) \triangleq det[\underline{A}_n s^n + \bar{A}_{n-1}s^{n-1} + \bar{A}_{n-2}s^{n-2} + \underline{A}_{n-3}s^{n-3} + \cdots],$$

$$\chi_3(s) \triangleq det[\underline{A}_n s^n + \underline{A}_{n-1}s^{n-1} + \bar{A}_{n-2}s^{n-2} + \bar{A}_{n-3}s^{n-3} + \cdots],$$

$$\chi_4(s) \triangleq det[\bar{A}_n s^n + \underline{A}_{n-1}s^{n-1} + \underline{A}_{n-2}s^{n-2} + \bar{A}_{n-3}s^{n-3} + \cdots].$$

Proof: Again, the "only if" part is trivial and the "if" part is shown as follows. Let s be a root of $\chi(s)$. Then

$$A_n s^n + A_{n-1}s^{n-1} + \cdots + A_1 s + A_0$$

is singular, and there exits unit vector v such that

$$[A_n s^n + A_{n-1}s^{n-1} + \cdots + A_1 s + A_0]A^{-1/2}v = 0.$$

Pre-multiplying by $v^* A^{-1/2}$ leads to

$$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

where $a_i \triangleq v^* A^{-1/2} A_i A^{-1/2} v$. It can be easily checked that $\bar{a}_i \geq a_i \geq \underline{a}_i$ for all i, and the rest of the proof follows from the scalar Kharitonov Theorem, since $\chi_i(s)$ reduce to the four Kharitonov polynomials similarly.□

This result cannot be further extended to general bounding matrices, as shown by a counterexample in section 5.

4. An Example

In this example we use the controller design procedure described in section 3 to examine the stability of the closed-loop system for various inertias. Consider the two-link planar manipulator with revolute joints and point masses at the distal end of the links as shown in Figure 1. The dynamics of the manipulator are given by

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta}) + G(\theta) = F \qquad (22)$$

where $F = [f_1 \;\; f_2]'$ is the vector of applied joint forces and

$$M(\theta) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \qquad (23)$$

is the inertia matrix with

$$m_{11} = (m_1 + m_2)\, l_1^2 + m_2\, l_2^2 + 2m_2 l_1 l_2 \cos(\theta_2)$$
$$m_{12} = m_{21} = m_2\, l_2^2 + m_2\, l_1\, l_2\, cos(\theta_2)$$
$$m_{22} = m_2\, l_2^2. \qquad (24)$$

$C(\theta, \dot{\theta}) = [c_1 \;\; c_2]'$ are centrifugal and Coriolis forces with

$$c_1 = -2\, m_2\, l_1\, l_2\, \dot{\theta}_1\, \dot{\theta}_2 \sin(\theta_2) - m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_2)$$
$$c_2 = m_2\, l_1\, l_2\, \dot{\theta}_1^2 \sin(\theta_2) \qquad (25)$$

and $G(\theta) = [G_1 \;\; G_2]'$ is the gravity force vector with

$$G_1 = (m_1 + m_2)\, g\, l_1\, cos(\theta_1) + m_2\, l_2\, g\, cos(\theta_1 + \theta_2)$$
$$G_2 = m_2\, g\, l_2\, cos(\theta_1 + \theta_2). \qquad (26)$$

Suppose the desired set point is $(\varepsilon, \theta, \dot{\theta}) = (0, \theta_d, 0)$, where $\varepsilon = \int[\theta(\tau) - \theta_d]d\tau$ as defined in section 1. Linearizing about this point gives $M(\theta_d)\ddot{\theta} = F$. Taking $m_1 = m_2 = 1$, $l_1 = l_2 = 1$, the inertia matrix simplifies to

$$M(\theta_d) = \begin{bmatrix} 3 + 2cos(\theta_2) & 1 + cos(\theta_2) \\ 1 + cos(\theta_2) & 1 \end{bmatrix}. \qquad (27)$$

An upper bound on the inertia matrix over a given range of θ_2 can be found with the help of the following lemma.

Lemma 5: Let M_1, M_2 be symmetric positive definite matrices. Let U be the transformation matrix such that $U'M_1U = \Sigma_1$ and $U'M_2U = \Sigma_2$ are diagonal. Let $\bar{\Sigma}$ be the diagonal matrix defined by $(\bar{\Sigma})_{ii} \triangleq \max((\Sigma_1)_{ii}, (\Sigma_2)_{ii})$. If $\bar{M} = U^{-'}\bar{\Sigma}U^{-1}$, then

$$M_1 \leq \bar{M}, \qquad M_2 \leq \bar{M} \qquad (28)$$

Furthermore, this $\bar{M}$ is the "smallest" to satisfy (28), in the sense that there is no other symmetric M such that

$$M_1 \leq M, \quad M_2 \leq M, \text{ and } M < \bar{M} \qquad (29)$$

Proof: The proof is straight-forward and hence omitted.□

This lemma can be used to generate a numerical upper bound on the family $M(\theta_d)$ by (1) discretizing the set, (2) choosing two members and finding an upper bound on these, (3) choosing another member and finding bound on the previous upper bound and this new member, etc.. The upper bound generated in this way will, in general, depend on the order that the elements are scanned, but can be used in the design procedure none-the-less.

Next, we design a robust controller based on an upper bounding inertia for the two-link manipulator. Suppose the desired set point θ_d has the property that $\theta_2 \in [0, \pi/2]$. For this particular example an upper bound $\bar{M}$ is generated using the above lemma just once with $M_1 = M(\theta_2 = 0)$ and $M_2 = M(\theta_2 = \pi/2)$. The upper bound is given by

$$\bar{M} = \begin{bmatrix} 5.06 & 1.85 \\ 1.85 & 1.35 \end{bmatrix}. \qquad (30)$$

If we place the closed-loop poles for each of the decoupled systems at $-1, -0.1+j, -0.1-j$, then the required controller gains are

$$K_D' = \begin{bmatrix} 1.20 & 0.00 \\ 0.00 & 1.20 \end{bmatrix},\ K_P' = \begin{bmatrix} 1.21 & 0.00 \\ 0.00 & 1.21 \end{bmatrix},\ K_I' = \begin{bmatrix} 1.01 & 0.00 \\ 0.00 & 1.01 \end{bmatrix}.$$

Using equation (15)

$$K_D = \begin{bmatrix} 6.07 & 2.22 \\ 2.22 & 1.62 \end{bmatrix},\ K_P = \begin{bmatrix} 6.12 & 2.24 \\ 2.24 & 1.63 \end{bmatrix},\ K_I = \begin{bmatrix} 5.11 & 1.87 \\ 1.87 & 1.36 \end{bmatrix}.$$

To check that the design is indeed stable for all $\theta_d \in [0, \pi/2]$ we compute the eigenvalues of the closed-loop system as a function of θ_2 and plot the stability degree of the system as a function of θ_d (see Figure 2). Note that the system is stable for $0 \le \theta_2 \le \pi/2$ (i.e. $M \le \bar{M}$) with a stability degree greater than the stability degree corresponding to $\bar{M}$. When θ_d is increased beyond $\pi/2$, increasing the inertia above $\bar{M}$, the stability degree decreases and the system (3) is eventually destabilized.

5. Counterexample

Following the positive results of section 3, it is natural to conjecture that Kharitonov's theorem may be extended to other polynomials with symmetric matrix coefficients lying in general matrix intervals. The answer is negative even for 3rd order polynomials as illustrated by the following counterexample. Let

$$K_D = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix},\ K_P = \begin{bmatrix} 5 & 5 \\ 5 & 9 \end{bmatrix},\ K_I = \begin{bmatrix} 4 & 5 \\ 5 & 12 \end{bmatrix}$$

$$\bar{M} = \begin{bmatrix} 6 & 7 \\ 7 & 9 \end{bmatrix},\ \underline{M} = \begin{bmatrix} 1 & 1 \\ 1 & 5/4 \end{bmatrix},\ M = \begin{bmatrix} 5/2 & 3 \\ 3 & 4 \end{bmatrix}.$$

It is easy to check that $\underline{M} < M < \bar{M}$ and that the equations

$$\chi(s) = det[\bar{M}s^3 + K_d s^2 + K_P s + K_I], \tag{31}$$

$$\chi(s) = det[\underline{M}s^3 + K_d s^2 + K_P s + K_I] \tag{32}$$

are Hurwitz. But

$$\chi(s) = det[M s^3 + K_d s^2 + K_P s + K_I]$$

is not. Hence it is not sufficient to check the extremal polynomials (31) and (32). Our result is therefore very particular to mechanical systems and our design procedure.

6. Conclusion

In this paper a simple sufficient condition for robust stability of a large class of PID-controlled mechanical systems was derived. This adds to the works of Shiel *et al.* [7] who found the conditions for stability of second order matrix polynomials. This is one of a few realistic applications of Kharitonov's theorem and serves as additional motivation for pursuing results of this kind. A procedure for designing a stabilizing controller is outlined and it is shown that a controller designed based on an upper bounding matrix stabilizes all other inertias. Furthermore, we have presented a matrix Kharitonov theorem for polynomials with interval matrix coefficients with all bounding matrices having the same shape. A counter example indicates that our matrix Kharitonov theorem cannot be extended to general bounding matrices.

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Figure 1.

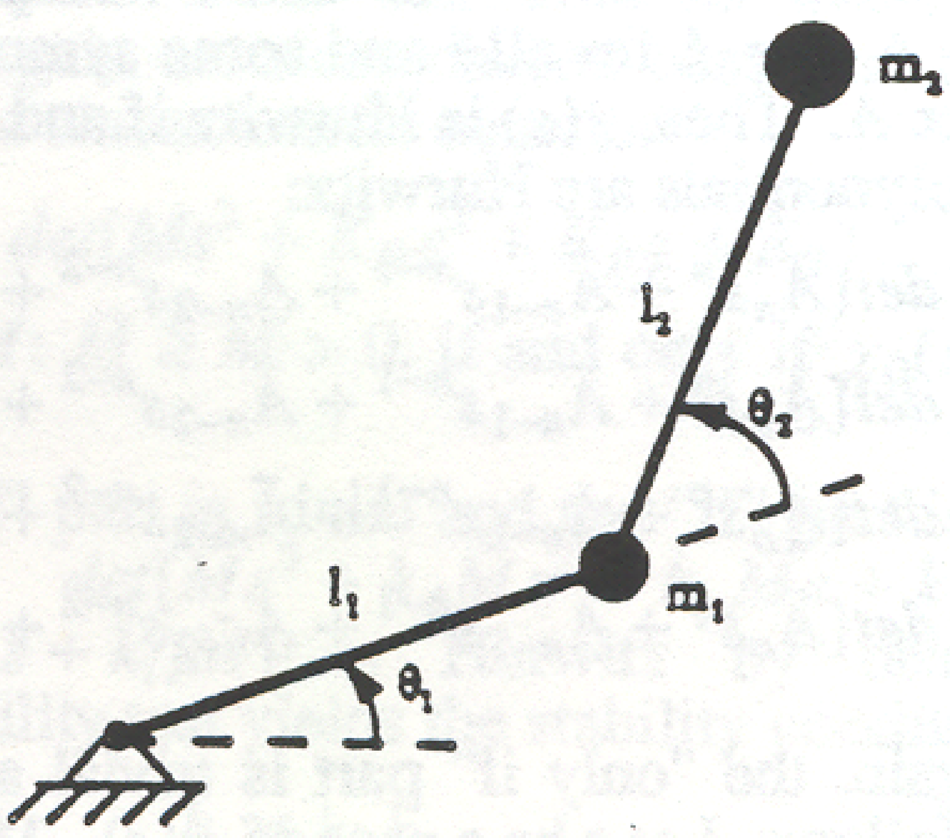

Figure 2.

Stability Curve - Nominal Design for $\theta_2 = \pi/2$