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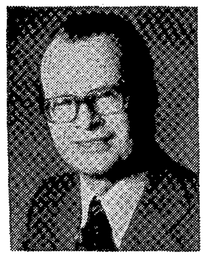

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Active Filters with Zero Amplifier Sensitivity

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***Abstract*—A general characterization of second-order active-*RC* filters employing one, two, and three operational amplifiers is given. When certain conditions are met, it is shown that the desired poles can be rendered insensitive to first- and second-order changes in the time constants of the operational amplifiers. Several novel circuits possessing this zero pole-sensitivity property, as well as zero ω_0 and Q sensitivity properties, are presented and discussed. Experimental verification of the results and comparisons to other popular second-order active-*RC* realizations bear out the significantly superior performance of these filters.**

Introduction

In recent years, a great deal of attention has been directed to designing active-*RC* second-order filters that have low sensitivities with respect to the active element. It is well known that the poles are displaced from their nominal positions because of the finite value of the gain-bandwidth product of the operational amplifier, hereafter designated by GB. In some circuits GB also affects the zeros. Although the dependence of the transfer function on GB has been extensively studied for a large number of filter circuits, no comprehensive discussion has been presented to discuss the effects of GB in a general way. Indeed, even for specific circuits, no consistent method for evaluating performance has been used.

In Part I of this paper, a general characterization of active-*RC* filters employing one, two, and three operational amplifier(s) (OA) is presented. It is demonstrated that it is possible to obtain zero pole sensitivities with respect to the operational amplifier for circuits employing two OA's. Zero pole sensitivity for each desired pole is sufficient to guarantee zero ω_0, Q, and transfer-function-magnitude sensitivities.

In Part II, circuits that possess the zero-sensitivity property derived in Part I are given. Expressions are then obtained that predict how the complex poles move when incremental changes in the operational amplifier are considered. Also, a novel two-input amplifier with a different gain function for each input is introduced and its usefulness in active filter design is discussed.

In Part III, a method of comparing the performance of high-Q filters due to incremental as well as infinitesimal changes in the characteristics of the OA is presented. Here, the superior frequency response of zero-sensitivity filters is clearly demonstrated. Furthermore, the location of the *parasitic poles* (the additional poles introduced by the nonideal OA's) is also investigated to determine whether they cause instability.

Part I—General Characterization

It is assumed that the OA has infinite input and zero output impedance, infinite common-mode rejection ratio, and that it is characterized by a single pole which is, for all practical purposes, at the origin (see [1] and [3]). The OA gain function is thus

$$A(s)=\frac{\mathrm{GB}}{s}=\frac{1}{s\tau} \tag{1}$$

where GB is the gain-bandwidth product in radians per second. The reciprocal of GB is designated by τ and is called the time constant of the OA. The OA can be rendered ideal by making GB infinite or τ zero.

Refer now to Fig. 1 where three OA's are used in conjunction with the *RC* network to realize a given transfer function. The equations for the three dependent variables are

Manuscript received May 3, 1977; revised February 22, 1978 and August 4, 1978.
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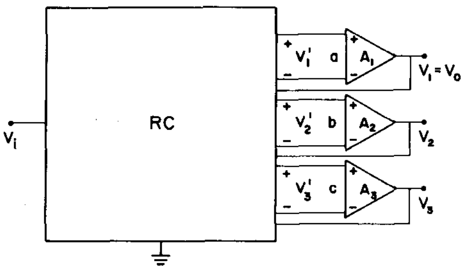


Fig. 1. General three-OA active RC filter.

$$
\begin{aligned}
\frac{V_1}{A_1} &= V_1T_{1'1}+V_2T_{1'2}+V_3T_{1'3}+V_iT_{1'i}\\
\frac{V_2}{A_2} &= V_1T_{2'1}+V_2T_{2'2}+V_3T_{2'3}+V_iT_{2'i}\\
\frac{V_3}{A_3} &= V_1T_{3'1}+V_2T_{3'2}+V_3T_{3'3}+V_iT_{3'i}
\end{aligned}
\tag{2}
$$

where for $\alpha\in\{1',2',3'\}$ and $x\in\{1,2,3,i\}$ the transfer function $T_{\alpha x}$ is defined in the usual sense as

$$
T_{\alpha x}=\left.\frac{V_\alpha}{V_x}\right|_{V_k=0}, \qquad \text{for } k\in\{1,2,3,i\}-\{x\}. \tag{3}
$$

Let V_1 represent the output of the circuit as indicated in Fig. 1. Then the transfer function of the circuit is given by

$$
\frac{V_0}{V_i}=\frac{\begin{vmatrix} T_{1'i} & -T_{1'2} & -T_{1'3}\\ T_{2'i} & \frac{1}{A_2}-T_{2'2} & -T_{2'3}\\ T_{3'i} & -T_{3'2} & \frac{1}{A_3}-T_{3'3}\end{vmatrix}}{\begin{vmatrix} \frac{1}{A_1}-T_{1'1} & -T_{1'2} & -T_{1'3}\\ -T_{2'1} & \frac{1}{A_2}-T_{2'2} & -T_{2'3}\\ -T_{3'1} & -T_{3'2} & \frac{1}{A_3}-T_{3'3}\end{vmatrix}}=\frac{\Delta_1}{\Delta} \tag{4}
$$

where

$$
\begin{aligned}
\Delta_1 = &-T_{1'i}(T_{2'2}T_{3'3}-T_{2'3}T_{3'2})\\
&-T_{2'i}(T_{1'3}T_{3'2}-T_{1'2}T_{3'3})\\
&-T_{3'i}(T_{1'2}T_{2'3}-T_{1'3}T_{2'2})\\
&-\frac{1}{A_2}(T_{1'3}T_{3'i}-T_{1'i}T_{3'3})\\
&-\frac{1}{A_3}(T_{1'2}T_{2'i}-T_{1'i}T_{2'2})\\
&-\frac{1}{A_2A_3}(T_{1'i})
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\Delta = &\,T_{1'1}(T_{2'2}T_{3'3}-T_{2'3}T_{3'2})\\
&+T_{2'1}(T_{1'3}T_{3'2}-T_{1'2}T_{3'3})\\
&+T_{3'1}(T_{1'2}T_{2'3}-T_{1'3}T_{2'2})\\
&+\frac{1}{A_1}(T_{2'3}T_{3'2}-T_{2'2}T_{3'3})\\
&+\frac{1}{A_2}(T_{1'3}T_{3'1}-T_{1'1}T_{3'3})\\
&+\frac{1}{A_3}(T_{1'2}T_{2'1}-T_{1'1}T_{2'2})\\
&+\frac{1}{A_1A_2}(T_{3'3})+\frac{1}{A_1A_3}(T_{2'2})\\
&+\frac{1}{A_2A_3}(T_{1'1})-\frac{1}{A_1A_2A_3}.
\end{aligned}
\tag{6}
$$

Since the various transfer functions used in (5) and (6) are associated with the same passive RC network, they all have a common denominator (characteristic) polynomial which in factored form is given by

$$
D_{RC}(s)=H(s+\alpha_1)(s+\alpha_2)\cdots \tag{7}
$$

where H is a constant and $0<\alpha_1<\alpha_2<\cdots$. In cases where pole-zero cancellation is possible in the expression for $T_{i'j}$, $T_{i'j}$ is not in lowest terms; for instance, if $T_{i'j}$ were a constant ($H_{i'j}$, say) then, the transfer function $T_{i'j}$ would be written as

$$
T_{i'j}=H_{i'j}\frac{D_{RC}(s)}{D_{RC}(s)}. \tag{8}
$$

A. Filters with One Operational Amplifier

In Fig. 1 let A_2 and A_3 be zero, thereby grounding the V_2 and V_3 outputs and let the RC network be second order. From (5) and (6) the transfer function becomes

$$
\frac{V_0}{V_i}=\frac{-T_{1'i}}{T_{1'1}-\frac{1}{A_1}}=\frac{-N_{1'i}}{N_{1'1}-\frac{D_{RC}}{A_1}} \tag{9}
$$

where the symbols N represent the numerator polynomials associated with the transfer functions T. For a second-order system with complex poles and for $A_1=1/s\tau_1$, (9) becomes

$$
\frac{V_0}{V_i}=\frac{+(a_2s^2+a_1s+a_0)}{s^2+s\frac{\omega_0}{Q}+\omega_0^2+s\tau_1D_{RC}(s)} \tag{10}
$$

where

$$
D_{RC}(s)=H(s+\alpha_1)(s+\alpha_2) \tag{11}
$$

and a_0,a_1,a_2 are constants determined by $N_{1'i}$.

The zeros of V_0/V_i are not affected by the time constant of the OA. The poles, however, are functions of τ_1 and are, therefore, displaced from their nominal ($\tau_1=0$)

positions. Let p_0 designate the desired upper half-plane pole. The Maclaurin's series expansion of this pole about its nominal position

$$p_0=-\frac{\omega_0}{2Q}+j\omega_0\sqrt{1-\frac{1}{4Q^2}}$$

is

$$p(\tau_1)=p_0+\left.\frac{\partial p}{\partial \tau_1}\right|_{\tau_1=0}\tau_1+\text{higher order terms.} \qquad (12)$$

The merit of the pole derivative with respect to the active element parameter τ_1 in calculating infinitesimal changes in the pole location is immediately apparent from (12). Using the characteristic equation obtained from (10), $\partial p/\partial \tau$ can be evaluated as follows:

$$p^2+p\frac{\omega_0}{Q}+\omega_0^2+p\tau_1 D_{RC}(p)=0 \qquad (13)$$

$$2p\frac{\partial p}{\partial \tau_1}+\frac{\omega_0}{Q}\frac{\partial p}{\partial \tau_1}+pD_{RC}(p)+\tau_1\left[D_{RC}(p)+p\frac{\partial D_{RC}(p)}{\partial p}\right]\frac{\partial p}{\partial \tau_1}=0. \qquad (14)$$

For $\tau_1=0$, (14) simplifies to

$$\left.\frac{\partial p}{\partial \tau_1}\right|_{\tau_1=0}=-\left[\frac{p_0}{2p_0+\frac{\omega_0}{Q}}\right]D_{RC}(p_0). \qquad (15)$$

to

$$\frac{V_0}{V_i}=\frac{-(T_{1'2}T_{2'i}-T_{1'i}T_{2'2})-\frac{1}{A_2}(T_{1'i})}{(T_{1'2}T_{2'1}-T_{1'1}T_{2'2})+\frac{1}{A_1}(T_{2'2})+\frac{1}{A_2}(T_{1'1})-\frac{1}{A_1A_2}}. \qquad (16)$$

Now let $A_1=1/s\tau_1$ and $A_2=1/s\tau_2$. For systems using a second-order RC network, (16) becomes

$$\frac{V_0}{V_i}=\frac{-\frac{(N_{1'i}N_{2'2}-N_{1'2}N_{2'i})}{D_{RC}}-s\tau_2 N_{1'i}}{\frac{(N_{1'1}N_{2'2}-N_{1'2}N_{2'1})}{D_{RC}}+s\tau_1(N_{2'2})+s\tau_2(N_{1'1})+s^2\tau_1\tau_2 D_{RC}} \qquad (17)$$

where $D_{RC}=H(s+\alpha_1)(s+\alpha_2)$.

The first OA affects only the poles whereas the second OA affects the poles and the zeros. To make the zeros independent of A_2, it is required that $N_{1'i}=0$; that is, in Fig. 1 the RC structure must not couple the input signal to port a while V_1 and V_2 are held at zero. With this constraint, (17) becomes

$$\frac{V_0}{V_i}=\frac{\frac{N_{1'2}N_{2'i}}{D_{RC}}}{\frac{(N_{1'1}N_{2'2}-N_{1'2}N_{2'1})}{D_{RC}}+s\tau_1(N_{2'2})+s\tau_2(N_{1'1})+s^2\tau_1\tau_2(D_{RC})}. \qquad (18)$$

The pole sensitivity is here defined to be the pole derivative (see Newcomb [2]). The normalization usually used in the sensitivity definition is not used here since the variable in the derivative ideally vanishes causing the normalization to vanish. The ω_0, Q, and transfer-function-magnitude sensitivities are likewise defined to be equal to the derivative of the respective functions. As (15) shows, the pole sensitivity depends upon the pole position and the characteristic polynomial of the RC network evaluated at the nominal pole position. Since $D_{RC}(s)$ becomes zero only for real values of s, $D_{RC}(p_0)$ cannot be zero if p_0 is assumed to have a nonzero imaginary part. Therefore, it follows for all single-OA second-order filters designed to realize a pair of complex conjugate poles that it is impossible to obtain zero pole sensitivity with respect to the time constant of the OA. At best one seeks realizations which result in a low value for $D_{RC}(p_0)$.

B. Filters with Two Operational Amplifiers

Setting $A_3=0$ in Fig. 1 and using (5) and (6), (4) reduces

Since the RC network is assumed to be second order, this equation can be expressed as

$$\frac{V_0}{V_i}=\frac{a_2s^2+a_1s+a_0}{\left(s^2+s\frac{\omega_0}{Q}+\omega_0^2\right)+s\tau_1(N_{2'2})+s\tau_2(N_{1'1})+s^2\tau_1\tau_2(D_{RC})}. \qquad (19)$$

The Maclaurin's series expansion of the desired upper half-plane pole about its nominal position is

$$p(\tau_1,\tau_2)=p_0+\frac{\partial p}{\partial \tau_1}\tau_1+\frac{\partial p}{\partial \tau_2}\tau_2+\frac{1}{2}\left[\frac{\partial^2 p}{\partial \tau_1^2}\tau_1^2+2\frac{\partial^2 p}{\partial \tau_1\partial \tau_2}\tau_1\tau_2+\frac{\partial^2 p}{\partial \tau_2^2}\tau_2^2\right]+\text{higher order terms.} \qquad (20)$$

where all derivatives are evaluated at the point (0,0). Using the denominator polynomial of (19), the first-order pole derivatives can be calculated as follows:

$$\left(p^2+p\frac{\omega_0}{Q}+\omega_0^2\right)+p\tau_1N_{2'2}(p) + p\tau_2N_{1'1}(p)+p^2\tau_1\tau_2D_{RC}(p)=0 \quad (21)$$

$$2p\frac{\partial p}{\partial\tau_1}+\frac{\omega_0}{Q}\frac{\partial p}{\partial\tau_1}+pN_{2'2}+\tau_1\left(N_{2'2}+p\frac{\partial N_{2'2}}{\partial p}\right)\frac{\partial p}{\partial\tau_1} + \tau_2\left(N_{1'1}+p\frac{\partial N_{1'1}}{\partial p}\right)\frac{\partial p}{\partial\tau_1} + \tau_2\left[p^2D_{RC}+\tau_1\left(2pD_{RC}+p^2\frac{\partial D_{RC}}{\partial p}\right)\frac{\partial p}{\partial\tau_1}\right]=0 \quad (22)$$

$$2p\frac{\partial p}{\partial\tau_2}+\frac{\omega_0}{Q}\frac{\partial p}{\partial\tau_2}+\tau_1\left(N_{2'2}+p\frac{\partial N_{2'2}}{\partial p}\right)\frac{\partial p}{\partial\tau_2} + pN_{1'1}+\tau_2\left(N_{1'1}+p\frac{\partial N_{1'1}}{\partial p}\right)\frac{\partial p}{\partial\tau_2} + \tau_1\left[p^2D_{RC}+\tau_2\left(2pD_{RC}+p^2\frac{\partial D_{RC}}{\partial p}\right)\frac{\partial p}{\partial\tau_2}\right]=0. \quad (23)$$

For $\tau_1=\tau_2=0$ and $p=p_0$, (22) and (23) simplify to

$$\left.\frac{\partial p}{\partial\tau_1}\right|_{\tau_1=\tau_2=0}=-\left[\frac{p_0}{2p_0+\frac{\omega_0}{Q}}\right]N_{2'2}(p_0) \quad (24)$$

$$\left.\frac{\partial p}{\partial\tau_2}\right|_{\tau_1=\tau_2=0}=-\left[\frac{p_0}{2p_0+\frac{\omega_0}{Q}}\right]N_{1'1}(p_0). \quad (25)$$

It is possible to obtain RC networks that have complex transmission zeros. For example, the second-order bridged-T network has this capability [3, p. 301]. Consequently, it is possible to make

$$N_{2'2}(p_0)=N_{1'1}(p_0)=0. \quad (26)$$

Therefore, it follows that in filters using two OA's, it is possible to obtain first-order pole sensitivities that are zero. In such filters, the pole displacements are due to second-order effects which can be calculated by considering second-order derivatives. Differentiating (22) and (23) and evaluating the results at the nominal pole position while requiring the first-order pole derivatives to vanish, the following results are obtained after some lengthy algebra:

$$\left.\frac{\partial^2 p}{\partial\tau_1^2}\right|_{\tau_1=\tau_2=0}=\left.\frac{\partial^2 p}{\partial\tau_2^2}\right|_{\tau_1=\tau_2=0}=0 \quad (27)$$

$$\left.\frac{\partial^2 p}{\partial\tau_1\partial\tau_2}\right|_{\tau_1=\tau_2=0}=-\frac{p_0^2}{2p_0+\frac{\omega_0}{Q}}D_{RC}(p_0). \quad (28)$$

Consequently, when $N_{2'2}(p_0)=N_{1'1}(p_0)=0$, the Maclaurin's series expansion for the desired pole simplifies to

$$p(\tau_1,\tau_2)=p_0-\frac{p_0^2}{2p_0+\frac{\omega_0}{Q}}D_{RC}(p_0)\tau_1\tau_2 + \text{higher order terms}. \quad (29)$$

The condition for zero first-order pole sensitivity for the desired pole (with nonzero imaginary part) as given by (26) can be obtained if $N_{1'1}$ and $N_{2'2}$ satisfy

a) $N_{1'1}(s)=0$, $N_{2'2}(s)=0$ (30)

b) $N_{1'1}(s)=0$, $N_{2'2}(s)=H_2\left(s^2+s\frac{\omega_0}{Q}+\omega_0^2\right)$ (31)

c) $N_{1'1}(s)=H_1\left(s^2+s\frac{\omega_0}{Q}+\omega_0^2\right)$, $N_{2'2}(s)=0.$ (32)

In these functions H_1 and H_2 are constants. The corresponding transfer functions in each of the three cases are

$$\frac{V_{0a}}{V_i}=\frac{N_{2'i}}{-N_{21}+s^2\tau_1\tau_2D_{RC}} =\frac{a_2s^2+a_1s+a_0}{\left(s^2+s\frac{\omega_0}{Q}+\omega_0^2\right)+s^2\tau_1\tau_2D_{RC}} \quad (33)$$

$$\frac{V_{0b}}{V_i}=\frac{N_{2'i}}{-N_{2'1}+s\tau_1N_{2'2}+s^2\tau_1\tau_2D_{RC}} =\frac{a_2s^2+a_1s+a_0}{\left(s^2+s\frac{\omega_0}{Q}+\omega_0^2\right)+s\tau_1H_2\left(s^2+s\frac{\omega_0}{Q}+\omega_0^2\right)+s^2\tau_1\tau_2D_{RC}} \quad (34)$$

$$\frac{V_{0c}}{V_i}=\frac{N_{2'i}}{-N_{2'1}+s\tau_2N_{1'1}+s^2\tau_1\tau_2D_{RC}} =\frac{a_2s^2+a_1s+a_0}{\left(s^2+s\frac{\omega_0}{Q}+\omega_0^2\right)+s\tau_2H_1\left(s^2+s\frac{\omega_0}{Q}+\omega_0^2\right)+s^2\tau_1\tau_2D_{RC}}. \quad (35)$$

For identical OA's, except for the H_1 and H_2 scale factors, there is no difference in the resulting characteristic polynomial when either (31) or (32) is satisfied. Although the set of conditions $N_{1'1}(s)=H_1(s^2+s(\omega_0/Q)+\omega_0^2)$ and $N_{2'2}(s)=H_2(s^2+s(\omega_0/Q)+\omega_0^2)$ also results in zero pole sensitivities, this case does not arise in second-order RC networks.

C. Filters with Three Operational Amplifiers

A study of (5) shows that A_2 and A_3 affect the zeros of V_0/V_i unless the following conditions are satisfied:

$$T_{1'i}=0, T_{1'2}T_{2'i}=0, T_{1'3}T_{3'i}=0. \quad (36)$$

Under these circumstances and for second-order systems with $A_1=1/s\tau_1$, $A_2=1/s\tau_2$, and $A_3=1/s\tau_3$, the transfer function becomes

$$\frac{V_0}{V_i}=\frac{a_2s^2+a_1s+a_0}{\left(s^2+s\frac{\omega_0}{Q}+\omega_0^2\right)+s\tau_1\frac{(N_{2'2}N_{3'3}-N_{2'3}N_{3'2})}{D_{RC}}+s\tau_2\frac{(N_{1'1}N_{3'3}-N_{1'3}N_{3'1})}{D_{RC}}+s\tau_3\frac{(N_{1'1}N_{2'2}-N_{1'2}N_{2'1})}{D_{RC}}+s^2\tau_1\tau_2N_{3'3}+s^2\tau_1\tau_3N_{2'2}+s^2\tau_2\tau_3N_{1'1}+s^3\tau_1\tau_2\tau_3D_{RC}}. \quad (37)$$

From the discussion presented in the two-amplifier case, it should be clear that first-order pole-sensitivities of a desired pole are zero if

$$\begin{aligned} &N_{2'2}(p_0)N_{3'3}(p_0)-N_{2'3}(p_0)N_{3'2}(p_0)=0\\ &N_{1'1}(p_0)N_{3'3}(p_0)-N_{1'3}(p_0)N_{3'1}(p_0)=0\\ &N_{1'1}(p_0)N_{2'2}(p_0)-N_{1'2}(p_0)N_{2'1}(p_0)=0. \end{aligned} \quad (38)$$

Furthermore, second-order pole sensitivities can also be made zero if in addition to (38)

$$N_{3'3}(p_0)=N_{2'2}(p_0)=N_{1'1}(p_0)=0. \quad (39)$$

When first- and second-order pole sensitivities are made zero, the Maclaurin's series expansion for the desired upper-half-plane pole becomes

$$p(\tau_1,\tau_2,\tau_3)=p_0+\frac{1}{3!}\left(\sum_{i=1}^{3}\tau_i\frac{\partial}{\partial\tau_i}\right)^3 p(\tau_1,\tau_2,\tau_3) + \text{higher-order terms} \quad (40)$$

where all derivatives are evaluated at the point (0,0,0).

It should be clear that zero sensitivities for the desired poles result in zero ω_0 and Q sensitivities. It can also be shown that the transfer-function-magnitude sensitivity is zero when the pole-sensitivity of each desired pole is zero [4].

Although the development presented here is for second-order RC systems, it can be extended to third-order RC systems which can be reduced to second order by pole-zero cancellations. Such a development would apply, for example, to twin-T RC networks. The zero-sensitivity development also extends to third- and higher order systems that realize higher order transfer functions.

It should be emphasized at this point that the zero-sensitivity conditions derived in this section do not guarantee either stability or realizability.

Part II—Circuits with Zero Pole Sensitivities

Many circuits exist that possess the zero-sensitivity properties derived in Part I. Three of these that are easily analyzed are introduced in this Section and others can be found in [4]. The OA related performance of these circuits may be considered representative of the zero-sensitivity designs, however no generalizations about zero-sensitivity designs in general are to be made in regard to component spread, passive sensitivity, etc.

Two second-order bandpass circuits with zero first-order pole-sensitivities are given in Fig. 2. For both circuits, $N_{1'1}(s)=0$ because with $A_2=0$, the output of OA1 cannot excite its own input port. Moreover, for the circuit of Fig. 2(a), $N_{2'2}(s)=0$ because with $A_1=0$, the output of OA2 cannot excite its own input port. For the circuit of Fig. 2(b), on the other hand, $N_{2'2}(s)=-1/2(s^2+s(1/Q)+1)$. As a result, it follows from (30) and (31) that the first-order derivatives of the desired poles with respect to the time constants of the OA's are zero for both circuits. Furthermore, for both circuits $N_{1'i}(s)=0$ because with $A_2=0$, the input voltage V_i does not excite the input port of OA1. From (17) it thus follows that the zeros of V_0/V_i are not dependent on the OA's.

The transfer functions of the circuits of Fig. 2(a) and (b) are given by

$$\frac{V_{01}}{V_i}=\frac{-2Qs}{\left(s^2+s\frac{1}{Q}+1\right)+s^2\tau_1\tau_2\left[s^2+s\left(\frac{1}{Q}+2Q\right)+1\right]} \quad (41)$$

$$\frac{V_{02}}{V_i}=\frac{-Qs}{\left(s^2+s\frac{1}{Q}+1\right)+\frac{s\tau_1}{2}\left(s^2+s\frac{1}{Q}+1\right)+s^2\tau_1\tau_2\left[s^2+s\left(\frac{1}{Q}+2Q\right)+1\right]}. \quad (42)$$

As τ_1 and τ_2 increase from zero, the desired poles begin to be displaced from their nominal positions while the parasitic poles move from infinity into the left half-plane.

Using (29), the changes in the desired upper-half-plane pole will now be calculated. Since D_{RC} is the same for both circuits, it follows from (29) that the second-order and lower effects of the desired pole displacements will be identical for small changes in the τs. For $\tau_1=\tau_2=\tau$, it follows that

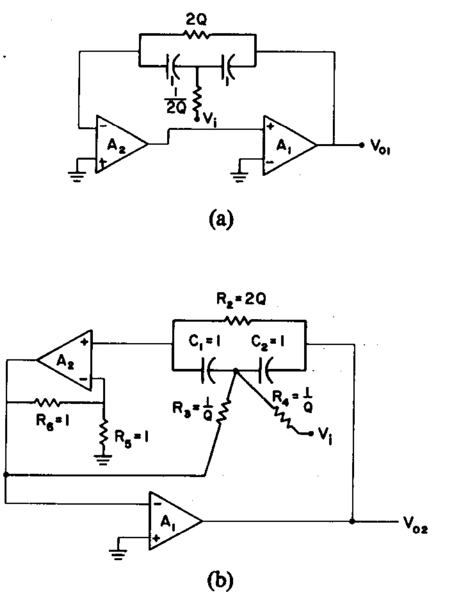


Fig. 2. Zero-sensitivity bandpass filters.

$$\Delta p \cong -\left[\frac{p_0^2}{2p_0+\dfrac{1}{Q}}\right] D_{RC}(p_0)\tau^2$$

$$= -\left[\frac{p_0^2}{2p_0+\dfrac{1}{Q}}\right]\left[p_0^2+p_0\left(\frac{1}{Q}+2Q\right)+1\right]\tau^2$$

$$= \left[\left(Q-\frac{1}{Q}\right)+j\frac{1}{2}\frac{\left(3-\dfrac{1}{Q^2}\right)}{\sqrt{1-\dfrac{1}{4Q^2}}}\right]\tau^2. \tag{43}$$

If the center frequency of the passband is shifted from 1 to ω_0, then the normalized change is given by

$$\frac{\Delta p}{\omega_0} \cong \left[\left(Q-\frac{1}{Q}\right)-j\frac{1}{2}\frac{\left(3-\dfrac{1}{Q^2}\right)}{\sqrt{1-\dfrac{1}{4Q^2}}}\right](\tau\omega_0)^2. \tag{44}$$

For high-Q circuits, the pole change can be approximated by

$$\frac{\Delta p}{\omega_0} \cong \left(Q-j\frac{3}{2}\right)(\tau\omega_0)^2 \cong Q(\tau\omega_0)^2. \tag{45}$$

Thus for $Q \geqslant 10$ the pole moves almost straight to the right. Indeed, if the pole-rate-of-change at the nominal position is maintained, both circuits will oscillate at the frequency $\omega_{0s}=\omega_0$ when

$$Q(\tau\omega_0)^2=\frac{1}{2Q}. \tag{46}$$

On the other hand, if ω_0 is restricted not to exceed 1 percent of GB, then $\Delta p/\omega_0 \leqslant 10^{-4}Q$; consequently, even for $Q=25$, the pole changes only 0.25 percent in magnitude. Furthermore, it is important to note that as long as the filter is stable, the Q of the pole increases as GB decreases from infinity but the ω_0 stays practically constant. As a result, the bandwidth of the bandpass characteristic decreases but the center frequency remains practically the same.

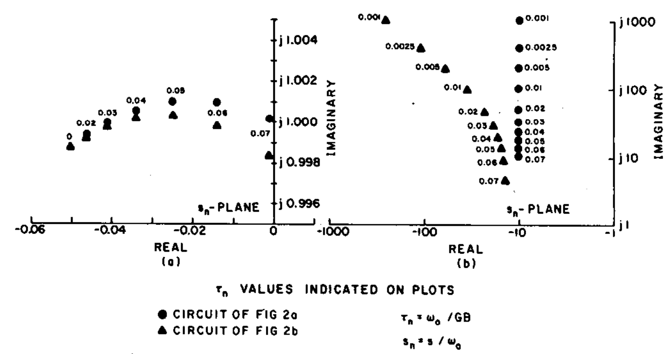


Fig. 3. Desired and parasitic poles for zero-sensitivity circuits of Fig. 2.

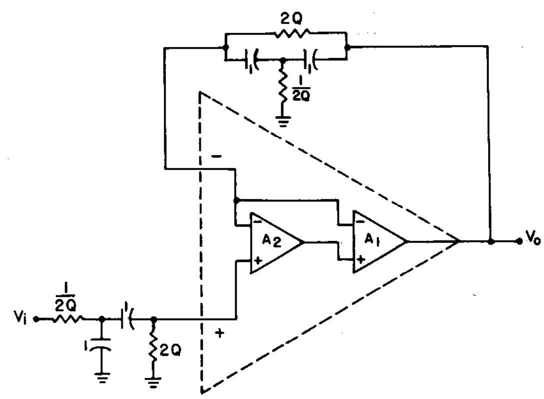


Fig. 4. Zero-sensitivity bandpass filter employing OA cluster.

To obtain a large scale picture of pole movement, the desired and parasitic upper-half-plane poles are plotted in Fig. 3 for $Q=10$ as a function of $\tau_n=(\omega_0/\mathrm{GB})$. It can be shown that regardless of the Q of the desired poles, the parasitic poles of circuit b have lower Q's than those of circuit a. Hence, circuit b is preferable. (Actually circuit a can be shown to be unstable for some OA's when the typical location of the second pole of the OA is included in the analysis.)

An additional zero-sensitivity second-order bandpass filter is shown in Fig. 4. An interesting property of this circuit is the fact that the two-amplifier cluster can be replaced with a single amplifier which has different gain functions for each of its two inputs. The transfer function is derived as follows:

$$(V_iT_iA^+ + V_0T_fA^-)=V_0 \tag{47}$$

where

$$T_i=\frac{2Qs}{s^2+s\left(\dfrac{1}{Q}+2Q\right)+1} \tag{48}$$

$$T_f = \frac{s^2 + s\frac{1}{Q} + 1}{s^2 + s\left(\frac{1}{Q} + 2Q\right) + 1} \tag{49}$$

$$A^+ = \frac{1}{s^3\tau_1\tau_2} \tag{50}$$

$$A^- = -\left(\frac{1}{s^2\tau_1\tau_2} + \frac{1}{s\tau_1}\right). \tag{51}$$

The V_0/V_i ratio then becomes

$$\frac{V_0}{V_i} = \frac{2Qs}{\left(s^2 + s\frac{1}{Q} + 1\right) + s\tau_2\left(s^2 + s\frac{1}{Q} + 1\right) + s^2\tau_1\tau_2\left[s^2 + s\left(\frac{1}{Q} + 2Q\right) + 1\right]}. \tag{52}$$

It follows from a comparison of (42) and (52) that the performance of the bandpass filters of Fig. 2(b) and Fig. 4 are very similar.

Although the zero-sensitivity realizations presented here result in bandpass transfer-functions, any realizable biquadratic transfer-function can be obtained with zero sensitivity by the appropriate modifications of the passive RC network of Fig. 1.

Part III—Comparisons and Experimental Results

The merit of any new circuit is best established by comparing it with accepted existing designs. In this part, the zero-sensitivity second-order bandpass circuit of Fig. 2(b) is compared with four popular existing second-order bandpass designs on a theoretical basis. The theoretical and experimental performance of the circuit of Fig. 4 is almost identical to that of the circuit of Fig. 2(b), so it has not been included in these comparisons.

Three different criteria are used to evaluate the performance of these circuits with respect to the time constants of the OA's.

The first evaluation is made on the basis of the sensitivity functions. In the case that the OA time constants are sufficiently small, the changes from nominal values of filter parameters are predictable from the infinitesimal changes obtained from a study of the sensitivity functions discussed earlier in this paper. Since the only circuit in this comparison that possesses zero-sensitivity properties is that of Fig. 2(b), the superiority of the zero-sensitivity design is apparent from this comparison.

The remaining two evaluations are made for values of $Q=10$ and 25 on the basis of the position of the desired poles and on the transfer-function-magnitude response using the time constants of the OA's as parameters. The comparisons show the actual pole positions and magnitude characteristics for a given OA time constant. Although the assumption of equal OA time constants is not necessary to obtain the zero-sensitivity conditions, it will be assumed for convenience in these comparisons that all OA's are identical.

The four existing circuits against which the zero-sensitivity design is compared are shown in Fig. 5. The first, introduced by Friend [5], uses only a single OA. The second, discussed by Sedra and Espinoza [6], is a two-OA bandpass filter. The commonly used state-variable filter, a form of which is shown in Fig. 5(c), uses three OA's; it appears in many texts and papers including [7]. The circuit of Fig. 5(d) which is noted for its low sensitivity properties is due to Mikhael and Bhattacharyya [8]. The transfer functions of these filters are also included in Fig. 5.

The center frequency of the passband has been shifted from 1 to ω_0 in the zero-sensitivity circuit of Fig. 2(b) in the comparisons that follow.

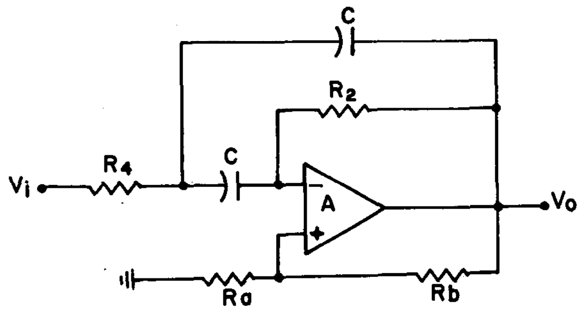
The desired pole loci for the four circuits of Fig. 5 and the zero-sensitivity circuit are shown in Fig. 6 for $Q=10$ and $Q=25$. To eliminate the center frequency as a parameter in the comparisons, the poles are plotted in the s_n-plane. The normalized frequency variable s_n is related to the frequency variable s by

$$s_n = \frac{s}{\omega_0}. \tag{53}$$

It can be seen that for the two-OA circuit of Fig. 2(b), the actual pole location is practically identical to the desired pole location for $\tau_n = \tau\omega_0 < 0.01$. It should be noted that one vertical unit on the graph represents about five horizontal units in Fig. 6(a) and ten horizontal units in Fig. 6(b). Therefore, the magnitude of the pole movement for Q's of 10 and 25 for the zero-sensitivity circuit is about an order of magnitude or more smaller than that for any of the other circuits compared for $\tau_n = 0.01$. As τ_n becomes smaller, the differences in the pole movement become even more pronounced. If the GB of an OA is $2\pi 10^6$ rad/s, a typical value for the 741, then the two-OA zero-sensitivity circuit should perform nearly as if the OA is ideal for a Q of 10 or 25 for center frequencies up to 10 kHz.

The complex conjugate parasitic poles of the five circuits under comparison are shown in Fig. 7. The parasitic poles of the circuits of Fig. 5(a) and Fig. 5(c) are on the negative real axis so do not appear in the figure. Although the parasitic poles for some of the circuits of Fig. 5 are more removed from the imaginary axis than those of the zero-sensitivity circuit, they are all nonetheless in the left half-plane. It is seen in the transfer-function-magnitude comparisons that the parasitic poles do not have any noticeable effect on the frequency response.

The normalized transfer-function magnitudes are compared in Figs. 8 and 9 for $Q=10$ and $Q=25$, respectively. The magnitude has been normalized so that the resonant frequency gain is unity for all circuits in the comparison



$$\omega_o = \sqrt{\frac{1}{R_2 R_4 C^2}}$$

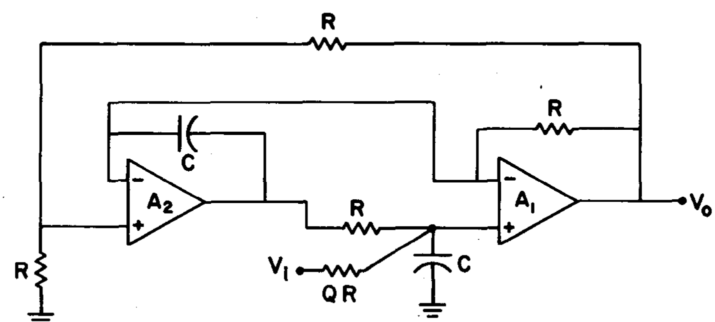
$$A = \frac{1}{\tau s}$$

$$h = \frac{R_2}{R_4} = 49$$

$$\frac{R_a}{R_b} = \frac{2}{h} + \frac{1}{Q\sqrt{h}}$$

$$T(s) = \frac{-s\,\omega_o\left[1 + \frac{2}{h} - \frac{1}{Q\sqrt{h}}\right]}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2 + \tau s\left[1 + \frac{2}{h} - \frac{1}{Q\sqrt{h}}\right]\left[s^2 + s\,\omega_o\left(\sqrt{h} + \frac{2}{\sqrt{h}}\right) + \omega_o^2\right]}$$

(a)



$$\omega_o = \frac{1}{RC}$$

$$A_1 = \frac{1}{\tau_1 s}$$

$$A_2 = \frac{1}{\tau_2 s}$$

$$T(s) = \frac{2s\frac{\omega_o}{Q} + \tau_2 2\frac{s}{Q}(s\omega_o + \omega_o^2)}{s^2 + s\frac{\omega_o}{Q} + \omega_o^2 + \tau_1 2s^2\left(s + \omega_o\left[1 + \frac{1}{Q}\right]\right) + \tau_2 2s\omega_o\left(s + \omega_o\left[1 + \frac{1}{Q}\right]\right) + \tau_1\tau_2 2s^2\left(s^2 + s\omega_o\left[2 + \frac{1}{Q}\right] + \omega_o^2\left[1 + \frac{1}{Q}\right]\right)}$$

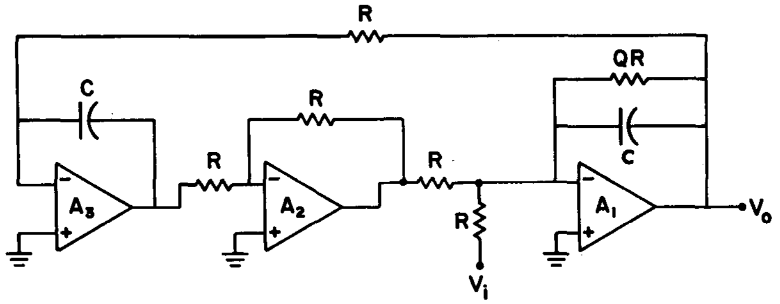
(b)

Fig. 5. Popular existing bandpass filters.

when the OA's are ideal. The normalized magnitude is thus obtained from the expression

$$\text{Normalized Magnitude} = \frac{|T(j\omega)|}{|T(j\omega_0)||_{\tau=0}}. \tag{54}$$

In Fig. 8 the magnitude response is plotted for values of $\tau_n = 0.01$, 0.025, and 0.05 and in Fig. 9 for values of $\tau_n = 0.005$ and 0.01. It is interesting to note that when the two-OA zero-sensitivity circuit does depart from the ideal, the center frequency remains nearly constant. This result could, of course, have been predicted from Fig. 6 since the desired pole movement is nearly horizontal for this circuit.



$$\omega_o = \frac{1}{RC}$$

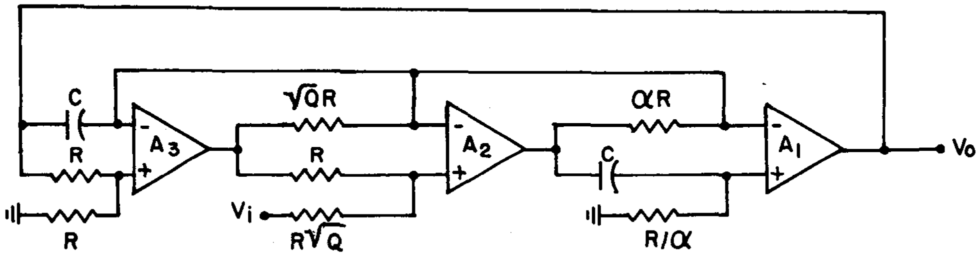
$$A_1 = \frac{1}{\tau_1 s}$$

$$A_2 = \frac{1}{\tau_2 s}$$

$$A_3 = \frac{1}{\tau_3 s}$$

$$T(s) = \frac{s\omega_o + \tau_2 2\omega_o s^2 + \tau_3\,\omega_o(s^2 + s\omega_o) + \tau_2\tau_3 s^2 2\omega_o(s + \omega_o)}{\begin{array}{l} s^2+s\frac{\omega_o}{Q} + \omega_o^2+\tau_1 s^2[s+\omega_o(2 + \frac{1}{Q})]+\tau_2 2s^2(s + \frac{\omega_o}{Q})+\tau_3 s[s^2+s\omega_o[1 + \frac{1}{Q}] + \frac{\omega_o^2}{Q}]+\tau_1\tau_2 2s^3(s+\omega_o[2 + \frac{1}{Q}]) \\ +\tau_1\tau_3 s^2[s^2+s\omega_o[3+ \frac{1}{Q}]+\omega_o^2[2+ \frac{1}{Q}]]+\tau_2\tau_3 2s^2[s^2+s\omega_o[1+ \frac{1}{Q}]+ \frac{\omega_o^2}{Q}]+\tau_1\tau_2\tau_3 2s^3[s^2+s\omega_o(3+ \frac{1}{Q})+\omega_o^2[2+ \frac{1}{Q}]] \end{array}}$$

(c)



$$\omega_o = \frac{1}{RC}\ , \ A_1 = \frac{1}{\tau_1 s}\ , \ A_2 = \frac{1}{\tau_2 s}\ , \ A_3 = \frac{1}{\tau_3 s}\ , \ \alpha = \frac{2\sqrt{Q}}{1+\sqrt{Q}}$$

$$T(s) = \frac{2s\frac{\omega_o}{Q} + \tau_3\,s\frac{2}{Q}[s^2\sqrt{Q} + s\,\omega_o - \sqrt{Q}\,\omega_o^2]}{\begin{array}{l} s^2+s\frac{\omega_o}{Q} + \omega_o+\tau_1 2s(s\frac{\omega_o}{Q} +\omega_o^2) + \tau_2 s[\frac{1}{Q} + \frac{1}{\sqrt{Q}}][s\omega_o+\alpha\omega_o^2] + \tau_1\tau_2 2s^2\alpha[\frac{1}{Q}+\frac{1}{\sqrt{Q}}][s\frac{\omega_o}{\alpha}+\omega_o^2] \\ + \tau_1\tau_3\, 2s^2[1+\frac{1}{\sqrt{Q}}][s\frac{\omega_o}{\alpha}+\omega_o^2] + \tau_2\tau_3\, 2s^3(1+\frac{1}{\sqrt{Q}})(s\omega_o+\alpha\omega_o^2) \\ + \tau_1\tau_2\tau_3\, 2s^3[1+\frac{1}{\sqrt{Q}}][s+\alpha\omega_o][s+\frac{\omega_o}{\alpha}+\frac{\omega_o}{\sqrt{Q}}] \end{array}}$$

(d)

Fig. 5. (*Continued*).

The superior performance of the two-OA zero-sensitivity circuit for small values of τ_n ($\tau_n < 0.025$ for $Q = 10$ and $\tau_n < 0.01$ for $Q = 25$) is apparent from the comparisons shown in Figs. 6, 8, and 9. For larger values of τ_n, the performance of all circuits in this comparison differs significantly from the ideal.

The limitations of the proposed configuration as well as those of the other circuits in the comparison can also be determined from Fig. 6. The poles of the zero-sensitivity circuit of Fig. 2(b) move quite rapidly towards the right half-plane for larger values of τ_n. It can be seen in this figure that the poles of the zero-sensitivity configuration enter the right half-plane for $\tau_n \cong 0.07$ when $Q = 10$ and $\tau_n \cong 0.027$ when $Q = 25$ rendering the filter unstable for larger values of τ_n. The new configuration is thus preferable only for sufficiently small values of τ_n.

Experimental Results

The two-OA zero-sensitivity filter of Fig. 2(b) was tested using two 741-type OA's with measured GB's of

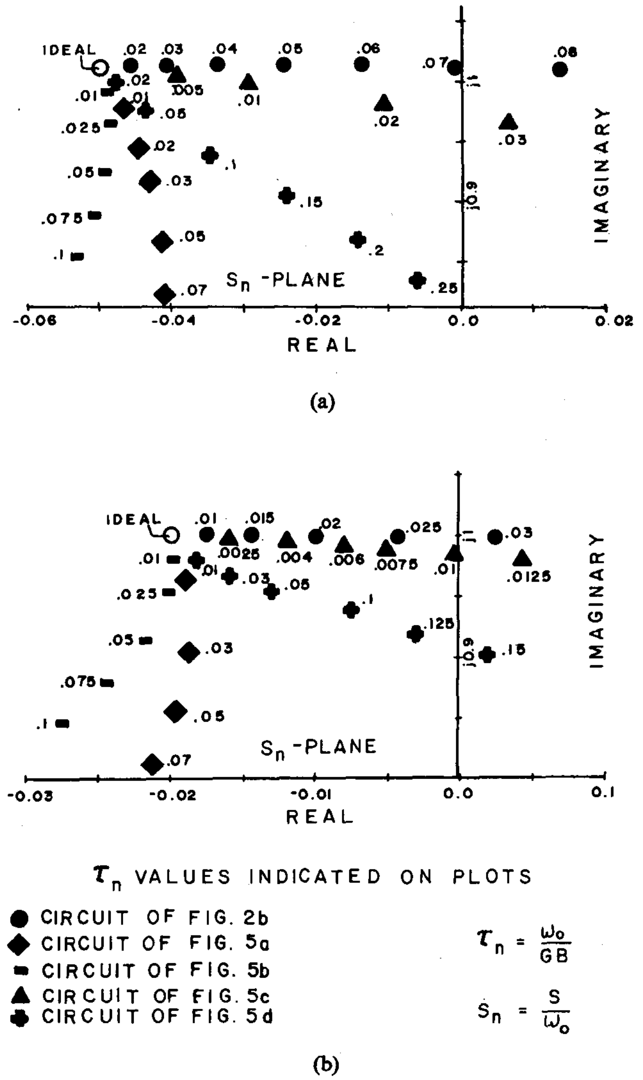


Fig. 6. Comparison of desired poles. (a) $Q=10$. (b) $Q=25$.

$5.4\times10^6 \pm 1$-percent rad/s. The bridged-T network was designed for a Q of 10 and center frequency of 10 kHz. The following measured component values were used in the design:

$$C_1 = C_2 = 1.23 \text{ nF}$$
$$R_3 \| R_4 = 686\ \Omega$$
$$R_2 = 261 \text{ k}\Omega$$
$$R_5 = 2.74 \text{ k}\Omega$$
$$R_6 = 2.74 \text{ k}\Omega. \quad (55)$$

With these component values, the theoretical and measured values of f_0 are, respectively, 9.65 and 9.634 kHz. The theoretical and measured values of Q are, respectively, 9.716 and 8.978.

Conclusion

A consistent method for comparing the performance of active filters with respect to the parameters of the OA based upon the active sensitivity function has been introduced. Constraints on the transfer function necessary to obtain zero sensitivity with respect to the parameters of all OA's in a filter were subsequently developed. These constraints were obtained without requiring either matched OA's or circuit components whose values were dependent upon the parameters of the OA.

Two practical circuits that satisfy the zero-sensitivity constraints were introduced to show that these constraints can be satisfied. The performance improvements obtained with these zero-sensitivity circuits were borne out by a detailed comparison of these circuits with popular existing designs.

Additional zero-sensitivity circuits can be designed to satisfy more stringent component spread and passive sensitivity requirements. Further, any realizable transfer function can be realized with a zero-sensitivity active filter.

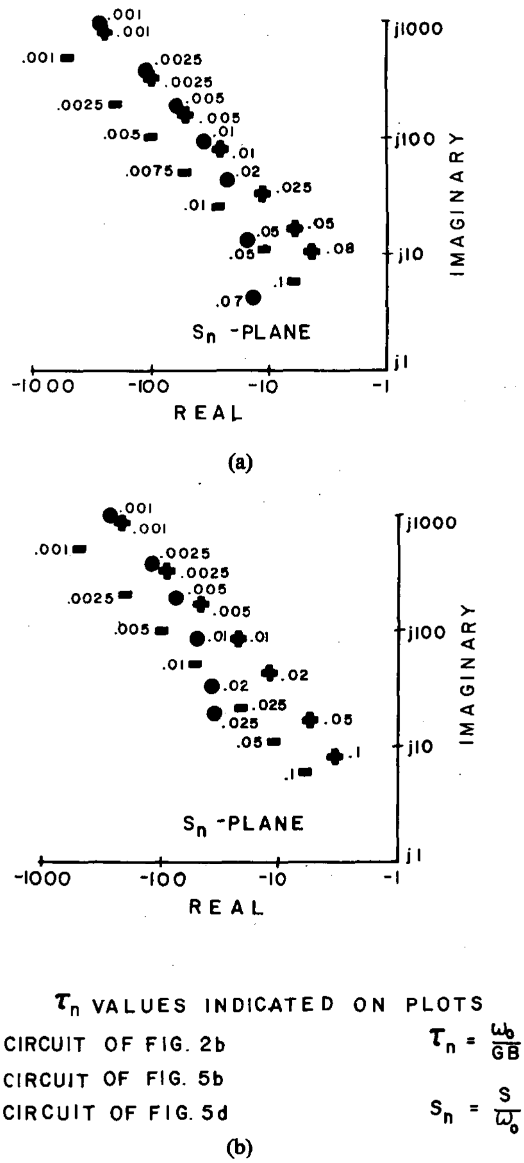


Fig. 7. Comparison of parasitic poles. (a) $Q=10$. (b) $Q=25$.

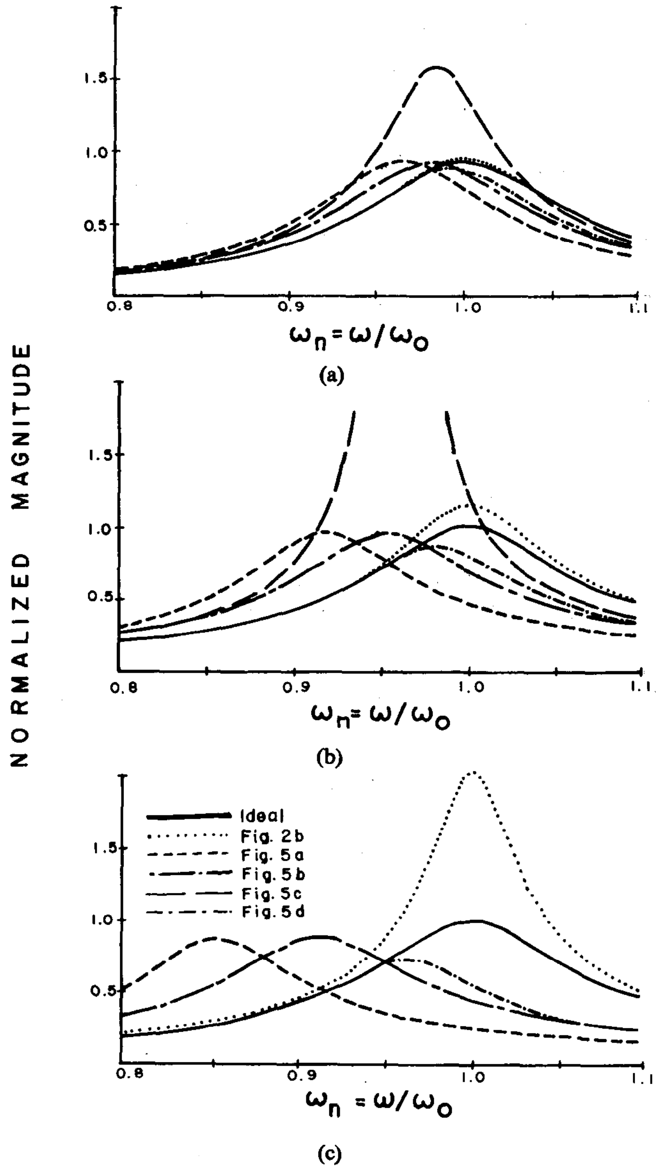


Fig. 8. Normalized magnitude response for $Q = 10$. (a) $\tau_n = 0.01$. (b) $\tau_n = 0.025$. (c) $\tau_n = 0.025$.

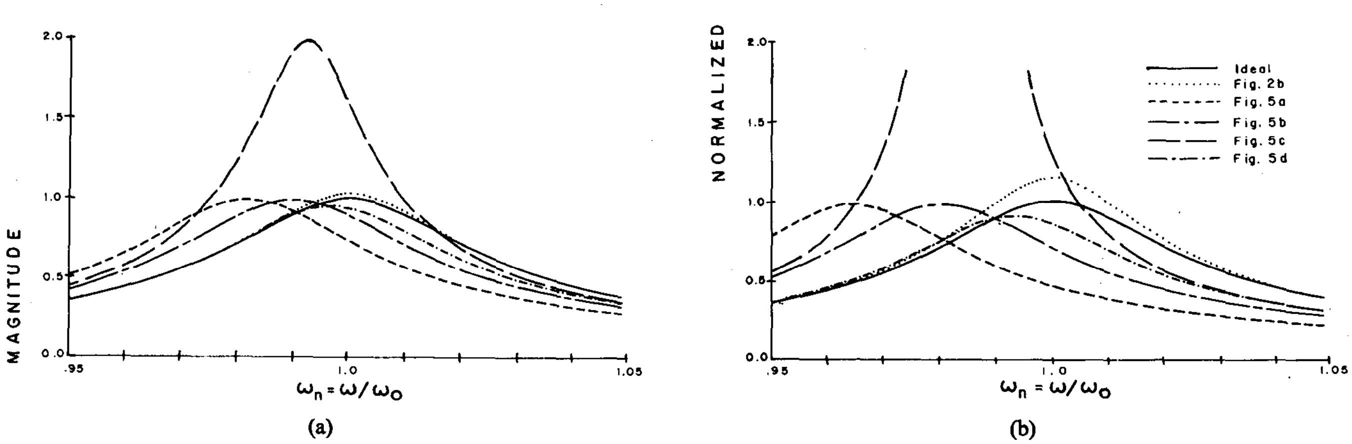


Fig. 9. Normalized magnitude response for $Q = 25$. (a) $\tau_n = 0.005$. (b) $\tau_n = 0.01$.

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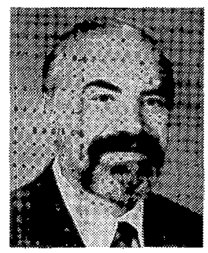

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Letters to the Editor

The x-Controlled Scalor and Its Applications to Network Synthesis

LIVIU GORAS

Abstract—**The problem of synthesizing linear or nonlinear x-controlled one-ports (where x can be a voltage or a current) is approached by introducing the linear x-controlled scalor, a scalor whose coefficients of the transmission matrix are proportional to some other electrical variables. In particular, any parametric network element can be obtained.**

The power scalor [1]–[3] (active transformer [4]) is a linear two-port element characterized by the following transmission matrix:

$$T^P = \begin{bmatrix} K_v & 0 \\ 0 & K_i \end{bmatrix} \tag{1}$$

where $K_v, K_i \in R - \{0\}$. Some particular cases are the voltage scalor (ideal voltage converter [5]) ($K_i = 1$), the current scalor (ideal current converter [5]) ($K_v = 1$), the power scalor with $K_v = K_i$ (ideal power converter [6], [7]) and, of course, the ideal transformer ($K_v = K_i = 1$).

Manuscript received March 28, 1978; revised September 6, 1978.
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The linear x-controlled power scalor (LXCPS) will be defined as a nonlinear four-port characterized by the following transmission matrix between ports 1 and 2:

$$T^P_{1-2(L)} = \begin{bmatrix} K_v x & 0 \\ 0 & K_i x' \end{bmatrix} \tag{2}$$

where x and x' can be currents or voltages applied at the remaining ports.

The linear x-controlled voltage scalor (LXCVS) and the linear x-controlled current scalor (LXCCS) are nonlinear three-ports characterized, respectively, by the transmission matrices between ports 1 and 2:

$$T^V_{1-2(L)} = \begin{bmatrix} K_v x & 0 \\ 0 & 1 \end{bmatrix} \qquad T^V_{1-2(L)} = \begin{bmatrix} 1 & 0 \\ 0 & K_i x' \end{bmatrix} \tag{3}$$

the controlling variables x and x' being applied at the third port.

Examples of LXCVS and LXCCS realizations are given in Fig. 1. The LXCPS can be obtained cascading a LXCVS with a LXCCS. Various other realizations are readily obtainable from [1, table III].