Control of Free-Flying Underactuated Space Manipulators to Equilibrium Manifolds

Ranjan Mukherjee, *Member, IEEE*, and Degang Chen, *Member, IEEE*

Abstract— **Underactuated mechanisms will provide low cost automation and will easily overcome actuator failures. These mechanisms will be particularly useful for space applications because of their reduced mass and lower power consumption. In space underactuation can be effectively introduced in robot manipulators. Such mechanisms will however be difficult to control because of the fewer number of actuators in the system. Of theoretical interest to us, is the problem where the unactuated joints do not have brakes. It is shown that in such a situation it is possible to bring the system to a complete rest and converge the actuated joints to their desired values, provided the system maintains zero momentum and none of the unactuated joints are cyclic coordinates. Our main interest is to converge both the actuated and the unactuated joints to their desired set of values. For this purpose, we assume that the number of actuated joints are more than the number of unactuated joints, and the unactuated joints have brakes. It is shown that if there exists sufficient dynamical coupling between the set of actuated and unactuated joints it is possible to converge all the manipulator joints to their desired values. In the paper, the orientation of the space vehicle is not controlled because this task can be accomplished after the manipulator has been reconfigured, using methods already developed.**

I. Introduction

A SPACE ROBOT mounted on a space vehicle is equipped with both internal-force actuators such as reaction wheels or control momentum gyroscopes, and external-force actuators such as jet thrusters to compensate for various disturbances. It is pragmatic to minimize the usage of the external-force actuators to maximize the useful life-span of the space robot. In situations where the space robot maintains zero momentum, it is possible to plan the motion of the system using the motion of the robot and without using the vehicle actuators [14], [22], or by using only two reaction wheels [7]. While vehicle actuators may be sparingly used, they cannot be eliminated from the design of space robotic systems for very practical reasons. Some of the joint actuators of a multilink space robotic system can however be eliminated for numerous advantages that we will discuss shortly.

We define an underactuated space robotic system as one that has fewer number of joint actuators than the number of its joints. Underactuation imposes second order nonholonomic constraints on the motion of the system. Only in particular situations, these second order differential constraints may be integrable into first order constraints [16]. Though underactuated dynamical systems are unconventional and will be difficult to control, such systems will have a number of advantages over completely actuated systems. Since the actuators of any dynamical system contribute largely to the cost of the system, underactuated mechanisms like robot manipulators will provide low cost automation. With fewer number of actuators, an underactuated mechanism will also be easier to design. The concept of underactuation can be extended to completely actuated mechanisms with actuator failures. Control strategies developed for underactuated systems will be useful in the event of actuator failures for robots on earth (limited to planar configurations) and more importantly in space where the repair or replacement of actuators will be a difficult task. These advantages have prompted us to investigate into the control of underactuated systems.

In 1991, the position control of a terrestrial manipulator composed of active and passive joints was discussed in [2]. The passive joints were assumed to have brakes instead of actuators. When the brakes were engaged, the passive joints were fixed and the active joints were controlled. When the brakes were released, the passive joints were indirectly controlled by the coupling characteristics of the manipulator dynamics. The position of the manipulator was controlled by engaging and disengaging the brakes. The kinematics and dynamics of underactuated manipulators was also studied in [6] where the spatial operator algebra was used to develop an algorithm for the inverse dynamics. The failure recovery control of space robotic systems was studied in [17].

A space vehicle that houses a completely actuated manipulator can be reoriented arbitrarily by using internal motion of the manipulator joints. It has been shown that in the planar case, the manipulator requires two actuated joints to achieve this task [12]. In the three dimensional case, the manipulator can perform this task with three actuated joints [4], [12], or two reaction wheels [7]. Using the same methods, an underactuated space manipulator can reorient its space vehicle, if it engages its brakes to fix the configuration of the actuated joints. Consequently, it is of practical importance to develop control strategies for the reconfiguration of all the joints of an underactuated manipulator. We discuss this problem in Section V-B. But first, we consider a problem that is of theoretical importance to us. We show in Section V-A that in the absence of brakes, it is possible to converge the actuated joints to their desired values and bring the system to a rest provided the system maintains zero momentum and none of the

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R. Mukherjee is with the Mechanical Engineering Department, Naval Postgraduate School, Monterey, CA 93943.

D. Chen is with the Electrical Engineering Department, Iowa State University, Ames, IA 50011.

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unactuated joints are cyclic coordinates. To prepare ourselves for discussion, we formulate the dynamics of underactuated space manipulators in Section II. In Section III, we discuss some of the issues related to the stability and controllability of our system, and in Section IV we discuss an asymptotic stability theorem that we use in Section V for developing control laws. Simulations are performed in Section VI to verify the efficacy of our control strategies developed in Section V.

II. Dynamics of Free-Flying Under-Actuated Systems—A Hamiltonian Formulation

In this section we formulate the dynamical equations of free-flying underactuated multibody systems in space. Without any loss of generality, the system is assumed to be an open chain of $(m + n)$ concatenated rigid bodies mounted on a space vehicle as shown in Fig. 1. We assume that out of these total $(m + n)$ joints, n are actuated. The generalized coordinates of the system consist of $\boldsymbol{q}_1 \in R^6$ representing the position and orientation (Euler angles) of the space vehicle, $\boldsymbol{q}_2 \in R^m$ representing the unactuated joint variables, and $\boldsymbol{q}_3 \in R^n$ representing the actuated joint variables. In space, the Lagrangian $L_0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is equivalent to the kinetic energy of the system, and is given as

$$L_0(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}} \tag{1}$$

$$\boldsymbol{M} \triangleq \begin{array}{c} \\ 6 \\ m \\ n \end{array} \begin{array}{c} \begin{array}{ccc} 6 & m & n \end{array} \\ \begin{pmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{M}_{13} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{M}_{23} \\ \boldsymbol{M}_{31} & \boldsymbol{M}_{32} & \boldsymbol{M}_{33} \end{pmatrix} \end{array}$$

$$\boldsymbol{q} \triangleq (\boldsymbol{q}_1^T \quad \boldsymbol{q}_2^T \quad \boldsymbol{q}_3^T)^T \in R^{(6+m+n)}$$

where $\boldsymbol{M} \in R^{(6+m+n)\times(6+m+n)}$ is the inertia matrix of the system and is a function of $\boldsymbol{q}$. Consequently, the dynamics of the system can be represented by the following vector equations:

$$\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{\boldsymbol{q}}_1}\right) - \left(\frac{\partial L_0}{\partial \boldsymbol{q}_1}\right) = 0 \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{\boldsymbol{q}}_2}\right) - \left(\frac{\partial L_0}{\partial \boldsymbol{q}_2}\right) = 0 \tag{3}$$

$$\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{\boldsymbol{q}}_3}\right) - \left(\frac{\partial L_0}{\partial \boldsymbol{q}_3}\right) = \boldsymbol{\tau}^T \tag{4}$$

where $\boldsymbol{\tau} \in R^n$ represents the vector of the joint torques at the actuated joints. The right hand side of (2) is zero because we do not use the reaction jets or the momentum wheels of the space vehicle to control the system.

An alternative way of depicting the motion of the space vehicle would be to consider the conservation of linear and angular momentum. Then, if the system maintains zero momentum, we can write instead of (2):

$$\dot{\boldsymbol{q}}_1 = \boldsymbol{H}\begin{pmatrix} \dot{\boldsymbol{q}}_2 \\ \dot{\boldsymbol{q}}_3 \end{pmatrix} \triangleq \boldsymbol{H}_1\, \dot{\boldsymbol{q}}_2 + \boldsymbol{H}_2\, \dot{\boldsymbol{q}}_3 \tag{5}$$

where the expression for $\boldsymbol{H}$ can be found in [14]. The above equation represents six first order differential constraints; three of these are integrable while the other three are nonholonomic

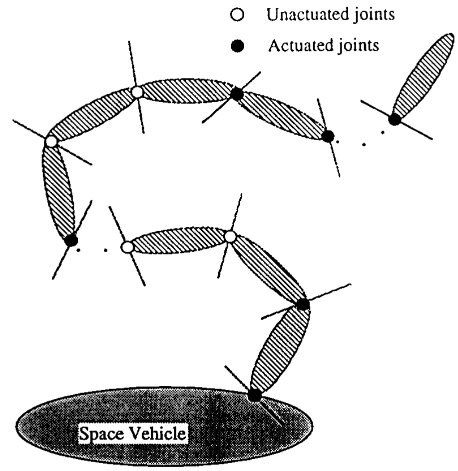

Fig. 1. A free-flying underactuated robot mechanism in space.

[14]. Equation (3) represents m differential constraints that include second order derivatives of the generalized coordinates, and are therefore second order nonholonomic constraints.

We use the transformation

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\tau}) = L_0(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{q}_3^T \boldsymbol{\tau} \tag{6}$$

to define the input dependent Lagrangian function $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\tau})$ [15]. Under this transformation, we have the following relations:

$$\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}_3}\right) = \left(\frac{\partial L_0}{\partial \dot{\boldsymbol{q}}_3}\right), \qquad \left(\frac{\partial L}{\partial \boldsymbol{q}_3}\right) = \left(\frac{\partial L_0}{\partial \boldsymbol{q}_3}\right) + \boldsymbol{\tau}^T. \tag{7}$$

By substituting (6) and (7) into (2), (3), and (4), we obtain the homogeneous dynamical equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \left(\frac{\partial L}{\partial \boldsymbol{q}}\right) = 0. \tag{8}$$

The generalized momentum $\boldsymbol{p} \in R^{(6+m+n)}$ corresponding to the generalized coordinates $\boldsymbol{q} \in R^{(6+m+n)}$ is defined by the relation

$$\boldsymbol{p} = \left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^T = \left(\frac{\partial L_0}{\partial \dot{\boldsymbol{q}}}\right)^T = \boldsymbol{M}\dot{\boldsymbol{q}}, \qquad \boldsymbol{p} \in R^{(6+m+n)}. \tag{9}$$

The input dependent *Hamiltonian function* $H(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{\tau})$ is next defined with the help of a Legendre transformation [5], as follows

$$H(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{\tau}) = \boldsymbol{p}^T \dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\tau}). \tag{10}$$

Using this transformation, we obtain from (8) the canonical equations

$$\dot{\boldsymbol{q}} = \left(\frac{\partial H}{\partial \boldsymbol{p}}\right)^T, \qquad \dot{\boldsymbol{p}} = -\left(\frac{\partial H}{\partial \boldsymbol{q}}\right)^T \tag{11}$$

Additionally, by substituting (6) in (10) we get the relation

$$H(\boldsymbol{q},\boldsymbol{p},\boldsymbol{\tau}) = H_0(\boldsymbol{q},\boldsymbol{p}) - \boldsymbol{q}_3^T\boldsymbol{\tau}, \qquad H_0(\boldsymbol{q},\boldsymbol{p}) \triangleq \boldsymbol{p}^T\dot{\boldsymbol{q}} - L_0(\boldsymbol{q},\dot{\boldsymbol{q}}) \tag{12}$$

which yields on differentiation

$$\dot{H}(\boldsymbol{q},\boldsymbol{p},\boldsymbol{\tau}) = \dot{H}_0(\boldsymbol{q},\boldsymbol{p}) - \dot{\boldsymbol{q}}_3^T\boldsymbol{\tau} - \boldsymbol{q}_3^T\dot{\boldsymbol{\tau}}$$

or

$$\left(\frac{\partial H}{\partial \boldsymbol{q}}\right)\dot{\boldsymbol{q}} + \left(\frac{\partial H}{\partial \boldsymbol{p}}\right)\dot{\boldsymbol{p}} + \left(\frac{\partial H}{\partial \boldsymbol{\tau}}\right)\dot{\boldsymbol{\tau}} = \dot{H}_0 - \dot{\boldsymbol{q}}_3^T\boldsymbol{\tau} - \boldsymbol{q}_3^T\dot{\boldsymbol{\tau}} \tag{13}$$

By substituting the relation $(\partial H/\partial \boldsymbol{\tau})^T = -\boldsymbol{q}_3$ from (12), and the canonical expressions of (11) in the above equation, we finally get

$$\dot{H}_0(q,p) = \dot{\boldsymbol{q}}_3^T\boldsymbol{\tau}. \tag{14}$$

To understand the physical significance of the previous equation, we take a look at the function H_0. Using (12), (1), and (9) we can show that

$$H_0(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{M}^{-1}\boldsymbol{p} = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}} = L_0(\boldsymbol{q},\dot{\boldsymbol{q}}) \tag{15}$$

where $\boldsymbol{M}^{-1}$ always exists because $\boldsymbol{M}$ is a positive definite matrix. The Hamiltonian function H_0 represents the kinetic energy or equivalently the total internal energy of the system. The physical significance of (14) is now clear. It implies that the rate of change of the internal energy of the system is equal to the external work done.

The Hamiltonian formulation of the dynamics has at least two advantages: 1) Since H_0 represents the total energy of the system, it can be used as a basis to construct Lyapunov functions. 2) The use of (14) helps us to plan the motion of the system in terms of the control input $\boldsymbol{\tau}$, thus eliminating the necessity of inverse dynamics computation.

III. Issues Related to Stability and Controllability

The simplest approach to study the controllability of a nonlinear system is to consider its linearization. If the linearized system is found to be controllable, the nonlinear system is controllable in the neighborhood of the equilibrium point. However the linearization approach is often unsatisfactory. In the process of linearization the nonlinear system may loose much of its structure. Therefore a nonlinear system may be controllable though its linearization may not. In our case, it is straightforward to show that the linearization of our system is not completely controllable.

The control of the wheeled mobile robot system was studied in [1]. The dynamic model was developed using a Lagrangian formalism and it was shown that static state feedback allows to reduce the dynamics of the system to a form where input-output linearizing control is possible. Such an analysis is particularly useful for simple nonholonomic systems.

The controllability of the rolling contact [10] and the single and multibody car systems [9] have been individually studied by constructing the control Lie algebra. For these systems the local controllability was ascertained by showing that the rank of the control Lie algebra is equal to the dimension of the state space at every point in the state space. Asymptotic stabilization using time-varying feedback was proposed in [21], and exponentially converging control laws were proposed in [20]. It should be emphasized that unlike most of these nonholonomic systems our system has a drift term. This is because of the presence of second order nonholonomic constraints that require the formulation of the problem at a dynamical level. Consequently, the analysis based on the control Lie algebra cannot be performed on our system.

In general our system may be asymptotically stabilizable by means of a linear or a nonlinear feedback. However, some necessary conditions was established in [3] for the existence of smooth (infinitely continuously differentiable) stabilizing feedback laws for the general nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}), \qquad \boldsymbol{x} \in R^N,\ \boldsymbol{u} \in R^M,\ \boldsymbol{f}(\boldsymbol{x}_e,0) = 0 \tag{16}$$

with $\boldsymbol{f}(.,.)$ continuously differentiable in the neighborhood of the equilibrium point $(\boldsymbol{x}_e, 0)$. One of the three conditions require the mapping

$$\gamma : R^N \times R^M \rightarrow R^N \qquad \text{defined by} \qquad \gamma : (\boldsymbol{x},\boldsymbol{u},) \rightarrow \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})$$

to be onto an open set containing the origin, where $\boldsymbol{x} = 0$. For our system, it can be shown that the mapping γ is not onto an open set containing the origin. Hence, there cannot exist a smooth feedback law that will stabilize the system to an equilibrium point. The objective of asymptotic stabilization might still be achievable by giving up the smoothness requirement for the feedback, or by stabilizing the system to an equilibrium manifold. In this paper we stabilize our system to equilibrium manifolds, instead of stabilizing it to an equilibrium point. Other examples of stabilization to equilibrium manifolds are [7], [14].

IV. Sufficient Conditions for Asymptotic Stability

The Lyapunov stability theorems provide sufficient conditions for proving the asymptotic stability of equilibrium points of dynamical systems. For autonomous systems these theorems are easy to apply when we can show that the derivative of the Lyapunov function is negative definite. When the derivative of the Lyapunov function is negative semidefinite, it may be possible to conclude the asymptotic stability of the equilibrium point using LaSalle's theorem [8], provided we can show that the maximum invariant set contains only the equilibrium point. It is straightforward to identify the set of points where the derivative of the Lyapunov function vanishes but the maximum invariant set is only a subset of this set. The main challenge of LaSalle's theorem is therefore to sort out the maximum invariant set.

In this section we discuss an asymptotic stability theorem [13] that provides us with sufficient conditions for proving the asymptotic stability of equilibrium points of autonomous systems when the first derivative of the Lyapunov function, which we assume to be analytic, is negative semidefinite. These sufficient conditions involve higher order derivatives of the Lyapunov function that contain the complete information of the dynamics of the system. Consequently, it becomes easier to identify the maximum invariant set.

Consider the nonautonomous system

$$\dot{x} = \boldsymbol{f}(t, \boldsymbol{x}) \tag{17}$$

where $\boldsymbol{f} : R_+ \times D \rightarrow R^n$ is a smooth vector field on $R_+ \times D, D \subset R^n$ is a neighborhood of the origin $\boldsymbol{x} = 0$. Let $\boldsymbol{x} = 0$ be an equilibrium point for the system described by (17). We then have

$$\boldsymbol{f}(t, 0) = 0, \qquad \forall t \geq 0 \tag{18}$$

Theorem: a) A necessary condition for stable nonautonomous systems

Let $V(t, \boldsymbol{x}) : R_+ \times D \rightarrow R_+$ be locally positive definite and analytic on $R_+ \times D$, such that

$$\dot{V}(t, \boldsymbol{x}) \triangleq \partial V/\partial t + (\partial V/\partial \boldsymbol{x})\boldsymbol{f}(t, \boldsymbol{x}) \tag{19}$$

is locally negative semidefinite. Then whenever an odd derivative of V vanishes, the next derivative necessarily vanishes and the second next derivative is necessarily negative semidefinite.

b) A sufficient condition for asymptotically stable autonomous systems.

Let $V(\boldsymbol{x}) : D \rightarrow R_+$ be locally positive definite and analytic on D, such that $\dot{V} \leq 0$. If there exists a positive integer k such that

$$\begin{cases} V^{(2k+1)}(\boldsymbol{x}) < 0 & \forall \boldsymbol{x} \neq 0 : \dot{\boldsymbol{V}}(\boldsymbol{x}) = 0 \\ V^{(i)}(\boldsymbol{x}) = 0 & \text{for } i = 2, 3, \cdots, 2k \end{cases} \tag{20}$$

where $V^{(*)}(\boldsymbol{x})$ denotes the $(*)$th time derivative of V with respect to time, then the equilibrium point is asymptotically stable. However, if $V^{(j)}(\boldsymbol{x}) = 0, \forall j = 1, 2, \cdots, \infty$, then the sufficient condition for the equilibrium point of the autonomous system to be asymptotically stable is that the set

$$S = \{\boldsymbol{x} : V^{(j)}(x) = 0, \forall j = 1, 2, \cdots, \infty\}$$

contains only the trivial trajectory $\boldsymbol{x} = 0$.

The proof of this theorem has been provided in the Appendix for reference.

V. Stabilization to Equilibrium Manifolds

In Section III we referred to a theorem [3], and showed that for our underactuated space manipulator there does not exist any smooth control law for feedback stabilization. In the next two subsections we discuss the asymptotic stabilization of our system to equilibrium manifolds.

A. Controlling the Actuated Joints Only

The state variables of our underactuated mechanism in space are denoted by $\boldsymbol{x} \triangleq (\boldsymbol{q}^T \;\; \boldsymbol{p}^T)^T$, where $\boldsymbol{q} \triangleq (\boldsymbol{q}_1^T \; \boldsymbol{q}_2^T \; \boldsymbol{q}_3^T)^T \in R^{6+m+n}$ denote the generalized coordinates and $\boldsymbol{p} \in R^{6+m+n}$ denote the corresponding generalized momenta. In this section we control the system such that $\boldsymbol{p} \rightarrow 0$, or equivalently $\dot{\boldsymbol{q}} \rightarrow 0$ from (9), and $\boldsymbol{q}_3 \rightarrow \boldsymbol{q}_{3d}$ at the final point of time; $\boldsymbol{q}_{3d}$ denotes the desired configuration of the actuated joints of the system. If such control can be established, the underactuated system would come to a complete rest with the actuated joints converging to their desired values simultaneously. We define a Lyapunov function [11] v as

$$v = H_0(\boldsymbol{q}, \boldsymbol{p}) + \frac{1}{2}\Delta\boldsymbol{q}_3^T \Delta\boldsymbol{q}_3, \qquad \Delta\boldsymbol{q}_3 \triangleq (\boldsymbol{q}_{3d} - \boldsymbol{q}_3) \tag{21}$$

where $H_0(\boldsymbol{q}, \boldsymbol{p})$ is the Hamiltonian of the system defined by (15). Clearly, $v = 0$ only on the equilibrium manifold $M_{E1} = \{\boldsymbol{x} : \boldsymbol{q}_3 = \boldsymbol{q}_{3d}, \boldsymbol{p} = 0\}$, and positive everywhere else. The derivative of v is computed as

$$\begin{aligned} \dot{v} &= \dot{H}_0 - \Delta\boldsymbol{q}_3^T \dot{\boldsymbol{q}}_3 \\ &= \dot{\boldsymbol{q}}_3^T \boldsymbol{\tau} - \Delta\boldsymbol{q}_3^T \dot{\boldsymbol{q}}_3 = \dot{\boldsymbol{q}}_3^T (\boldsymbol{\tau} - \Delta\boldsymbol{q}_3) \end{aligned} \tag{22}$$

where $\dot{H}_0 = \dot{\boldsymbol{q}}_3^T \boldsymbol{\tau}$ was substituted from (14). We choose $\boldsymbol{\tau}$ in (22) as

$$\boldsymbol{\tau} = \Delta\boldsymbol{q}_3 - \beta\dot{\boldsymbol{q}}_3, \tag{23}$$

where β is a positive constant. This results in

$$\dot{v} = -\beta\|\dot{\boldsymbol{q}}_3\|^2. \tag{24}$$

Clearly, $\dot{v}$ is negative semidefinite and is equal to zero if and only if $\dot{\boldsymbol{q}}_3 = 0$. At this point we can use LaSalle's theorem [8] to conclude the asymptotic stability of the equilibrium point only if we can show that the maximum invariant set of the superset $\{\boldsymbol{x} : \dot{\boldsymbol{q}}_3 = 0\}$ comprises only of the equilibrium manifold M_{E1}. However, there is no systematic way to sort out the maximum invariant set and therefore LaSalle's theorem will not be useful. In this situation we refer to the asymptotic stability theorem that was stated in the last section.

We begin by computing the higher order derivatives of the analytic Lyapunov function v defined by (21). We realize that when $\dot{v} = 0$, or equivalently $\dot{\boldsymbol{q}}_3 = 0$, we have $\ddot{v} \triangleq v^{(2)} = 0$ and $v^{(3)} = -2\beta\|\ddot{\boldsymbol{q}}_3\|^2 \leq 0$. Now if additionally $v^{(3)} = 0$, then we have $\dot{\boldsymbol{q}}_3 = \ddot{\boldsymbol{q}}_3 = 0$. This implies $v^{(4)} = 0$ and $v^{(5)} = -6\beta\|\boldsymbol{q}_3^{(3)}\|^2 \leq 0$, where $\boldsymbol{q}_3^{(3)}$ is the third derivative of $\boldsymbol{q}_3$ with respect to time. In other words, whenever an odd derivative of the Lyapunov function v vanishes the next derivative also vanishes and the second next derivative is found to be negative semidefinite. This is in complete agreement with the necessary conditions of our asymptotic stability theorem.

From the above discussion it easily follows that the choice of the control vector $\boldsymbol{\tau}$ in (23) results in

$$\begin{aligned} &v^{(2k+1)} = -\beta_k\|\boldsymbol{q}_3^{(k+1)}\|^2 \text{ for} \\ &\beta_k > 0 \;\text{ for } k = 1, 2, \cdots, \infty \end{aligned} \tag{25}$$

when $v^{(i)} = 0$ for $i = 1, 2, \cdots, 2k$. Therefore, when $\dot{v} = 0$ or equivalently $\dot{\boldsymbol{q}}_3 = 0$, if $\boldsymbol{q}_3^{(k+1)} \neq 0$ for some positive integer k, then the sufficient conditions of our theorem given by (20) are satisfied and we can conclude asymptotic stability of the equilibrium manifold M_{E1} of our system.

Let us now investigate the situation where $v^{(j)} = 0, \forall j = 1, 2, \cdots, \infty$. This implies from (25) that $\boldsymbol{q}_3^{(j)} = 0, \forall j = 1, 2, \cdots, \infty$, and therefore $\boldsymbol{q}_3$ will remain constant for all future times. Let this constant value of $\boldsymbol{q}_3$ be given as

$$\boldsymbol{q}_3 = \boldsymbol{q}_{3c}. \tag{26}$$

Equation (26) implies $\dot{\boldsymbol{q}}_3 = 0$, which on substitution in (5) results in

$$\dot{\boldsymbol{q}}_1 = \boldsymbol{H}_1 \dot{\boldsymbol{q}}_2. \tag{27}$$

Expanding (2), and then substituting (26) and (27), we get

$$\boldsymbol{M}_{11}\ddot{\boldsymbol{q}}_1 + \boldsymbol{M}_{12}\ddot{\boldsymbol{q}}_2 + \boldsymbol{N}_1(\dot{\boldsymbol{q}}_2) = 0$$

where $\boldsymbol{N}_1$ is the sum of all the centrifugal and coriolis terms. Since $\boldsymbol{M}_{11}$ is positive definite, we have

$$\ddot{\boldsymbol{q}}_1 = -\boldsymbol{M}_{11}^{-1}\{\boldsymbol{M}_{12}\ddot{\boldsymbol{q}}_2 + \boldsymbol{N}_1\} = 0. \tag{28}$$

Expanding (3), and then substituting (26) and (27), we get

$$\boldsymbol{M}_{21}\ddot{\boldsymbol{q}}_1 + \boldsymbol{M}_{22}\ddot{\boldsymbol{q}}_2 + \boldsymbol{N}_2(\dot{q}_2) = 0$$

where $\boldsymbol{N}_2$ is the sum of all the centrifugal and coriolis terms. By substituting (28) in the above equation we get

$$\widetilde{\boldsymbol{M}}_{22}\ddot{q}_2 - \boldsymbol{M}_{21}\boldsymbol{M}_{11}^{-1}\boldsymbol{N}_1 + \boldsymbol{N}_2 = 0,$$
$$\widetilde{\boldsymbol{M}}_{22} \triangleq (\boldsymbol{M}_{22} - \boldsymbol{M}_{21}\boldsymbol{M}_{11}^{-1}\boldsymbol{M}_{12}).$$

Since the $(6+m) \times (6+m)$ upper left sub-matrix of the matrix $\boldsymbol{M}$ in (1) is positive definite, the matrix $\widetilde{\boldsymbol{M}}_{22}$ in the above equation is always nonsingular. Hence, the motion of the unactuated joints can be expressed as

$$\ddot{\boldsymbol{q}}_2 = \widetilde{\boldsymbol{M}}_{22}^{-1}\{\boldsymbol{M}_{21}\boldsymbol{M}_{11}^{-1}\boldsymbol{N}_1 - \boldsymbol{N}_2\}. \tag{29}$$

Equation (29) is a set of m second order differential constraint on $\boldsymbol{q}_2$. Equation (4) tells us that the torque $\boldsymbol{\tau}$, as given by (23), maintains all the actuated joint angles $\boldsymbol{q}_3$ at the constant value $\boldsymbol{q}_{3c}$. Expanding (4) and then substituting (23), (26), (27), we get the following n equations

$$\boldsymbol{M}_{31}\ddot{\boldsymbol{q}}_1 + \boldsymbol{M}_{32}\ddot{\boldsymbol{q}}_2 + \boldsymbol{N}_3(\dot{\boldsymbol{q}}_2) = (\boldsymbol{q}_{3d} - \boldsymbol{q}_{3c})$$

where $\boldsymbol{N}_3$ is the sum of all the centrifugal and coriolis terms. By substituting (28) into the above equation we get

$$\left(\boldsymbol{M}_{32} - \widetilde{\boldsymbol{M}}_{31}\boldsymbol{M}_{12}\right)\ddot{\boldsymbol{q}}_2 - \widetilde{\boldsymbol{M}}_{31}\boldsymbol{N}_1 + \boldsymbol{N}_3 = (\boldsymbol{q}_{3d} - \boldsymbol{q}_{3c}),$$
$$\widetilde{\boldsymbol{M}}_{31} \triangleq \boldsymbol{M}_{31}\boldsymbol{M}_{11}^{-1}. \tag{30}$$

By substituting (29) into (30) we finally get

$$\left(\boldsymbol{M}_{32} - \widetilde{\boldsymbol{M}}_{31}\boldsymbol{M}_{12}\right)\widetilde{\boldsymbol{M}}_{22}^{-1}\Big\{\boldsymbol{M}_{21}\boldsymbol{M}_{11}^{-1}\boldsymbol{N}_1(\dot{\boldsymbol{q}}_2)$$
$$-\boldsymbol{N}_2(\dot{\boldsymbol{q}}_2)\Big\}\widetilde{\boldsymbol{M}}_{31}\boldsymbol{N}_1(\dot{\boldsymbol{q}}_2) + \boldsymbol{N}_3(\dot{\boldsymbol{q}}_2) = (\boldsymbol{q}_{3d} - \boldsymbol{q}_{3c}). \tag{31}$$

Equation (31) represents n first order differential constraints on $\boldsymbol{q}_2$.

Since the actuated joints are stationary at the configuration $\boldsymbol{q}_3 = \boldsymbol{q}_{3c}$, the motion of the system can be completely described by (27) and (29). The six first order differential constraints of (27) describe the motion of the space vehicle, and the m second order differential constraints of (29) describe the motion of the unactuated joints. Additionally, the n first order differential constraints given by (31) also describe the motion of the unactuated joints. When the number of unactuated joints are less than the number of actuated joints $(m < n)$, this implies that the m second order differential constraints of (29) and the n first order differential constraints given by (31) have the same solution. Of course, then the n first order differential constraints of (31) are not all independent.

The constraint equations given by (29) and (31) can have the same solution if and only if (29) is partially integrable into equations that are linear combinations of (31) or they both have the same trivial solution $\ddot{\boldsymbol{q}}_2 = \dot{\boldsymbol{q}}_2 = 0$. The necessary and sufficient condition for the partial integrability [16] of a second order differential constraint into a first order differential constraint is that the corresponding joint should be a cyclic coordinate [5], i.e., the inertia matrix $\boldsymbol{M}$ should not be a function of the particular joint angle. If some of the unactuated joints are cyclic, then the motion of the unactuated joints are at least partially represented by linear combinations of (31). In such a situation we cannot conclude $\dot{\boldsymbol{q}}_2 = 0$ since $\boldsymbol{q}_{3d} \neq \boldsymbol{q}_{3c}$. Therefore, the maximum invariant set $S = \{\boldsymbol{x} : v^{(j)}(\boldsymbol{x}) = 0, \forall j = 1, 2, \cdots, \infty\}$ would contain other trajectories along with the trivial trajectory $\boldsymbol{x} \triangleq (\Delta\boldsymbol{q}_3^T\ \boldsymbol{p}^T)^T = 0$. Then the asymptotic stability of the equilibrium manifold M_{E1} of our underactuated system cannot be guaranteed.

We now assume that none of the unactuated joints are cyclic coordinates and the number of actuated joints are greater than the number of unactuated joints $(n < m)$. In that case, the m second order homogeneous nonintegrable differential constraints given by (29) and the m linearly independent equations among the n first order nonhomogeneous differential equations given by (31) would have the same solution if and only if $\ddot{\boldsymbol{q}}_2 = \dot{\boldsymbol{q}}_2 = 0$. This would imply that $\boldsymbol{q}_{3c} = \boldsymbol{q}_{3d}$ and $\dot{\boldsymbol{q}}_1 = 0$ from (31) and (27) respectively. It would logically follow that the maximum invariant set $S = \{\boldsymbol{x} : v^{(j)}(\boldsymbol{x}) = 0, \forall j = 1, 2, \cdots, \infty\}$ would contain only the trivial trajectory $\boldsymbol{x} \triangleq (\Delta\boldsymbol{q}_3^T\ \ \boldsymbol{p}^T)^T = 0$. Hence our system would asymptotically stabilize to the equilibrium manifold M_{E1}. We summarize the results obtained so far in the form of a proposition.

Proposition: The underactuated space manipulator system described in Section II can always be stabilized to the equilibrium manifold $M_{E1} = \{\boldsymbol{x} : \boldsymbol{q}_3 = \boldsymbol{q}_{3d}, \boldsymbol{p} = 0\}$ using a control of the form as in (23), provided the number of actuated joints are greater then the number of unactuated joints, the initial momentum of the system is zero, and none of the unactuated joints are cyclic coordinates.

At this point it becomes necessary to mention that if the system contains no unactuated joints, or if the unactuated joints are held fixed using brakes, the control in (23) can still be used to converge the actuated joints to their desired values and bring the system to a rest.

B. Controlling All the Joints of the Manipulator

In this section we converge all the joints of the manipulator to their desired configuration and simultaneously bring the system to a rest. In other words, we control the system such that the system stabilizes to the equilibrium manifold $M_{E2} = \{\boldsymbol{x} : \boldsymbol{q}_2 = \boldsymbol{q}_{2d}, \boldsymbol{q}_3 = \boldsymbol{q}_{3d}, \boldsymbol{p} = 0\}$. We assume that the unactuated joints are provided with brakes, the number of actuated joints is greater than the number of unactuated joints, there exists sufficient dynamical coupling between the

TABLE I
KINEMATIC AND DYNAMIC PARAMETERS OF SPACE ROBOT USED IN SIMULATION

	Mass	Inertia	Length
Vehicle	10.14	0.208200	r =0.20
Link-1	1.55	0.013690	l_1 =0.32
Link-2	1.35	0.009225	l_2=0.28
Link-3	1.21	0.006665	l_3 =0.25

actuated and the unactuated joints, and the system maintains zero momentum. We achieve our goal by first converging the unactuated joints to their desired values by exploiting the dynamical coupling between the actuated and the unactuated joints. The brakes of the unactuated joints are then engaged and the underactuated manipulator behaves as a completely actuated system. Subsequently, the convergence of the actuated joints to their desired values and the dissipation of the kinetic energy of the system can be achieved by using the control law developed in Section V-A under zero momentum conditions.

To first converge the unactuated joints to their desired values, we define the vector $\boldsymbol{s}$

$$\boldsymbol{s} = \Delta\ddot{\boldsymbol{q}}_2 + k\,\Delta\dot{\boldsymbol{q}}_2 + \Delta\boldsymbol{q}_2, \qquad \Delta\boldsymbol{q}_2 \triangleq (\boldsymbol{q}_{2d} - \boldsymbol{q}_2) \tag{32}$$

where k is a positive constant and $\boldsymbol{q}_{2d}$ is the desired configuration of the unactuated joints. The elements of the vector $\boldsymbol{s}$ are s, and $s = 0$ denotes a sliding surface[1] [19] corresponding to a particular unactuated joint. Each sliding surface $s = 0$ can be seen to have the globally asymptotically stable equilibrium point $q_2 = q_{2d}$. If the condition $\boldsymbol{s} = 0$ can be maintained, then the convergence of all the unactuated joints to their desired values will be guaranteed.

Since $\boldsymbol{q}_{2d}$ is constant, we have from the definition of $\boldsymbol{s}$

$$\boldsymbol{s} = -\ddot{\boldsymbol{q}}_2 - k\dot{\boldsymbol{q}}_2 + \Delta\boldsymbol{q}_2. \tag{33}$$

Equation (3) can be expanded into the form

$$\boldsymbol{M}_{22}\ddot{\boldsymbol{q}}_2 = -(\boldsymbol{M}_{21}\ddot{\boldsymbol{q}}_1 + \boldsymbol{M}_{23}\ddot{\boldsymbol{q}}_3 + \boldsymbol{N}_U) \tag{34}$$

where $\boldsymbol{N}_U$ is the sum of all the centrifugal and coriolis forces at the unactuated joint. Multiplying (33) by the positive definite matrix $\boldsymbol{M}_{22}$ and then substituting the expression for $\boldsymbol{M}_{22}\ddot{\boldsymbol{q}}_2$ from (34) we have

$$\boldsymbol{M}_{22}\boldsymbol{s} = \boldsymbol{M}_{21}\ddot{\boldsymbol{q}}_1 + \boldsymbol{M}_{23}\ddot{\boldsymbol{q}}_3 + \boldsymbol{N}_U - \boldsymbol{M}_{22}(k\,\dot{\boldsymbol{q}}_2 - \Delta\boldsymbol{q}_2). \tag{35}$$

In this equation, if we choose $\ddot{\boldsymbol{q}}_3$ of the form

$$\ddot{\boldsymbol{q}}_3 = -\boldsymbol{M}_{23}^{\#}(\boldsymbol{M}_{21}\ddot{\boldsymbol{q}}_1 - \boldsymbol{M}_{22}(k\,\dot{\boldsymbol{q}}_2 - \Delta\boldsymbol{q}_2) + \boldsymbol{N}_U) \tag{36}$$

where $\boldsymbol{M}_{23}^{\#}$ is the pseudoinverse of $\boldsymbol{M}_{23}$, we have

$$\begin{aligned}\boldsymbol{M}_{22}\boldsymbol{s} &= (\boldsymbol{E}_m - \boldsymbol{M}_{23}\boldsymbol{M}_{23}^{\#})(\boldsymbol{M}_{21}\ddot{\boldsymbol{q}}_1 - \boldsymbol{M}_{22}(k\,\dot{\boldsymbol{q}}_2 - \Delta\boldsymbol{q}_2) + \boldsymbol{N}_U)\\ &= (\boldsymbol{E}_m - \boldsymbol{M}_{23}\boldsymbol{M}_{23}^{\#})(\boldsymbol{M}_{21}\ddot{\boldsymbol{q}}_1 - \boldsymbol{M}_{22}(k\,\dot{\boldsymbol{q}}_2 - \Delta\boldsymbol{q}_2)\\ &\quad + \boldsymbol{M}_{23}\ddot{\boldsymbol{q}}_3 + \boldsymbol{N}_U)\\ &= (\boldsymbol{E}_m - \boldsymbol{M}_{23}\boldsymbol{M}_{23}^{\#})\boldsymbol{M}_{22}(-\ddot{\boldsymbol{q}}_2 - k\,\dot{\boldsymbol{q}}_2 + \Delta\boldsymbol{q}_2)\\ &= (\boldsymbol{E}_m - \boldsymbol{M}_{23}\boldsymbol{M}_{23}^{\#})\boldsymbol{M}_{22}\boldsymbol{s}\end{aligned} \tag{37}$$

[1] A sliding surface is defined as a surface such that all trajectories on this surface converge to the globally asymptotically stable equilibrium point on the surface.

where we used the identity $(\boldsymbol{E}_m - \boldsymbol{M}_{23}\boldsymbol{M}_{23}^{\#})\boldsymbol{M}_{23} = 0$, and substituted (33) and (34). This equation implies that the choice of $\ddot{\boldsymbol{q}}_3$ as given in (36) results in

$$\boldsymbol{M}_{23}\boldsymbol{M}_{23}^{\#}\boldsymbol{M}_{22}\boldsymbol{s} = 0. \tag{38}$$

We assume that there exists sufficient dynamical coupling between the actuated and the unactuated joints such that $\boldsymbol{M}_{23} \in R^{m\times n}$ is full rank everywhere except possibly at a few discrete singular points in the workspace. Then (38) implies $\boldsymbol{s} = 0$ except at the singularities. This is true because, a) when $\boldsymbol{M}_{23} \in R^{m\times n}$ is full rank, and $m \leq n, \boldsymbol{M}_{23}\boldsymbol{M}_{23}^{\#}$ is equal to the identity matrix of size m, and b) $\boldsymbol{M}_{22}$ is always nonsingular. Since $\boldsymbol{s} = 0$ implies global asymptotic convergence of all the unactuated joints to their desired values, we conclude that it is possible to converge the unactuated joints to their desired values if the inertial coupling matrix between the actuated and the unactuated joints is full rank.

Ideally, we would carry out the inverse dynamics computation to obtain the torque $\boldsymbol{\tau}$ that produces the acceleration as given by (36). This would require significant amount of computation. Since the torque $\boldsymbol{\tau}$ has a direct causal relationship with $\ddot{\boldsymbol{q}}_3$, we choose the torque $\boldsymbol{\tau}$ simply proportional to the acceleration $\ddot{\boldsymbol{q}}_3$, i.e.,

$$\boldsymbol{\tau} = -C\boldsymbol{M}_{23}^{\#}[\boldsymbol{M}_{21}\ddot{q}_1 - \boldsymbol{M}_{22}(k\,\dot{\boldsymbol{q}}_2 - \Delta\boldsymbol{q}_2) + \boldsymbol{N}_U], \qquad C > 0 \tag{39}$$

where C is a constant of proportionality. In the above equation we need to use acceleration feedback, which is rather unconventional. Since the inertia terms of the space vehicle is larger than that of the robot manipulator, the acceleration $\ddot{\boldsymbol{q}}_1$ will be quite small as compared to $\ddot{\boldsymbol{q}}_3$ in (36). Consequently, we simplify (39) as

$$\boldsymbol{\tau} = C\boldsymbol{M}_{23}^{\#}[\boldsymbol{M}_{22}(k\,\dot{\boldsymbol{q}}_2 - \Delta\boldsymbol{q}_2) - \boldsymbol{N}_U]. \tag{40}$$

We are not tracking any particular trajectory of $\boldsymbol{q}_2$, instead, our goal is to converge the unactuated joints to their desired values. Hence, the modified feedback law in (40) is expected to work as efficiently as (39). In the next section we show that our expectations are fulfilled.

To converge all the joints to their desired configuration, we will first use the control input given by (40). This will take the unactuated joints to their desired values and the unactuated joint velocities to zero. The brakes at the unactuated joints will then be engaged and the system will behave as a completely actuated system. Consequently, the control input given by (23) can be used to converge the actuated joints to their desired values and simultaneously dissipate the energy from the system. Based on our discussion in Section V-A, we know that the control input in (23) asymptotically converges the system to the manifold M_{E1}. But since we have $\boldsymbol{q}_2 = \boldsymbol{q}_{2d}$, we actually converge to the manifold M_{E2} which is a submanifold of M_{E1}.

VI. SIMULATIONS

Simulations were done with a planar 3-DOF space robot having revolute joints, as shown in Fig. 2. One of the joints of the robot was left unactuated. The kinematic and the dynamic

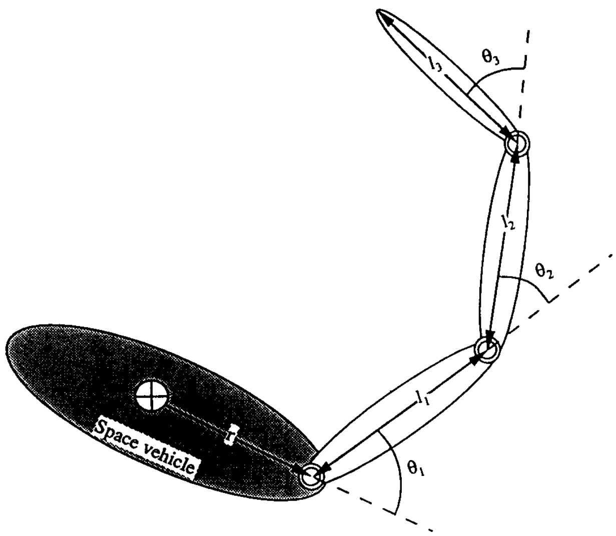

Fig. 2. A planar three-link underactuated space manipulator system.

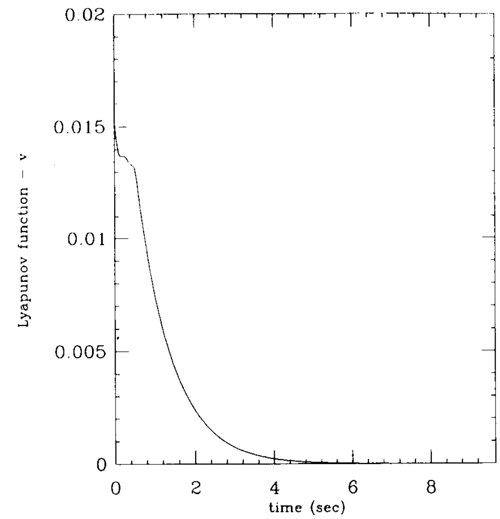

Fig. 3. Variation of the Lyapunov function with time for the simulation in Section V-A.

parameters of the robot, assumed to be made of aluminum, were chosen according to Table I, where S.I. units were used. The inertia of the vehicle and the links of the robot were measured about reference frames located at their individual center of the masses. The parameters r, l_1, l_2, and l_3 are defined in Fig. 2.

Controlling the Actuated Joints Only: It can be easily shown by constructing the inertia matrix of the system that none of the three joint angles of the manipulator are cyclic coordinates. We opted to leave the first joint of the manipulator unactuated. For this planar case, $\boldsymbol{q}_1 \in R^3$ represents the two Cartesian coordinates of the center of mass of the vehicle and the orientation of the vehicle in the plane, and $q_2 \in R$ and $\boldsymbol{q}_3 \in R^2$ represent the unactuated and the actuated joint angles respectively. The initial system configuration (refer to Fig. 2) was assumed to be

$$\begin{aligned}\boldsymbol{Q}_i &\triangleq (q_2 \quad \boldsymbol{q}_3^T \quad \dot{q}_2 \quad \dot{\boldsymbol{q}}_3^T)\\ &= (\theta_1 \quad \theta_2 \quad \theta_3 \quad \dot{\theta}_1 \quad \dot{\theta}_2 \quad \dot{\theta}_3)\\ &= (-15.0 \quad 0.0 \quad 10.0 \quad 0.0 \quad 0.0 \quad 0.0)\end{aligned}$$

where the joint angles are in degrees, the joint velocities in degrees/sec, and the Cartesian position in meters.

The desired configuration of the actuated joints in degrees were $\boldsymbol{q}_{3d} \triangleq (0.0 \quad 0.0)^T$. The control law in (23) was used with $\beta = 1.75$. The computation was terminated as the value of the Lyapunov function reduced below 1×10^{-6}. The convergence time was noted to be 9.65 s. Fig. 3 shows the variation of the Lyapunov function with time. From the figure, it is understood that at around $t = 0.2$ s, the derivative of the Lyapunov function momentarily goes to zero. It subsequently becomes negative and guarantees asymptotic stability as predicted by the Theorem discussed in Section IV. Fig. 4 shows the trajectory of all the joints of the manipulator. At the final point of time, both the actuated joints converge to their desired configuration. The unactuated joint is not controlled in this situation. From Fig. 4 it is clear that the velocity of all the joints approach zero at the final time and the system comes to a complete rest.

Controlling All the Joints of the Manipulator: In this particular simulation we assumed that the second joint of the manipulator was unactuated. The initial and desired configurations of the system (refer to Fig. 2) were assumed to be

$$\begin{aligned}\boldsymbol{Q}_i &\triangleq (q_2 \quad \boldsymbol{q}_3^T \quad \dot{q}_2 \quad \dot{\boldsymbol{q}}_3^T)\\ &\triangleq (\theta_2 \quad \theta_1 \quad \theta_3 \quad \dot{\theta}_2 \quad \dot{\theta}_1 \quad \dot{\theta}_3)\\ &= (0.0 \quad 15.0 \quad 45.0 \quad 0.0 \quad 0.0 \quad 0.0)\\ \boldsymbol{Q}_f &= (0.0 \quad 15.0 \quad 45.0 \quad 30.0 \quad 15.0 \quad 45.0).\end{aligned}$$

where the joint angles are in degrees, the joint velocities in degrees/sec, and the Cartesian position in meters.

We used a value of $k = 2.0$ to define the vector of the sliding surfaces $\boldsymbol{s}$ in (32). The value of C in (39) was chosen to be 1.0, and we used a convergence criterion of 0.001 radians for the unactuated joint. In Fig. 5, where the trajectories of all the joints of the manipulator have been plotted, at around $t = 12.37$ seconds, this criterion was satisfied. Subsequently, the brake at the unactuated joint was engaged. The control law was changed from (40) to (23), and the computation was terminated at $t = 19.45$ seconds when the Lyapunov function reduced below 0.000025.

VII. Conclusion

An underactuated manipulator is one that has fewer number of joint actuators than the number of links of the manipulator. If such a manipulator is properly controllable, it will have a number of advantages over a similar completely actuated system. Such systems are more feasible for space applications due to the absence of gravity. Also, they prove to be more useful for space applications. In this paper we discussed the

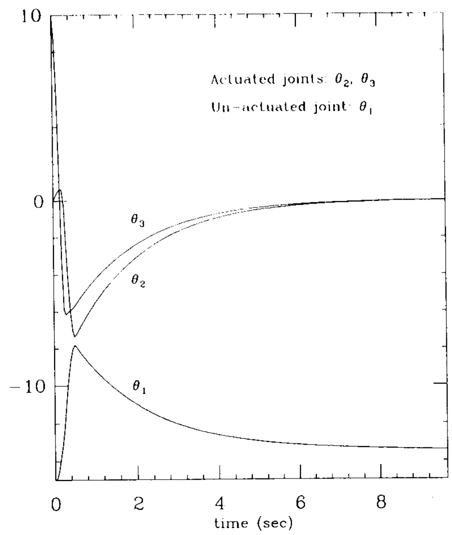

Fig. 4. Trajectories of all the joints of the manipulator for the simulation in Section V-A.

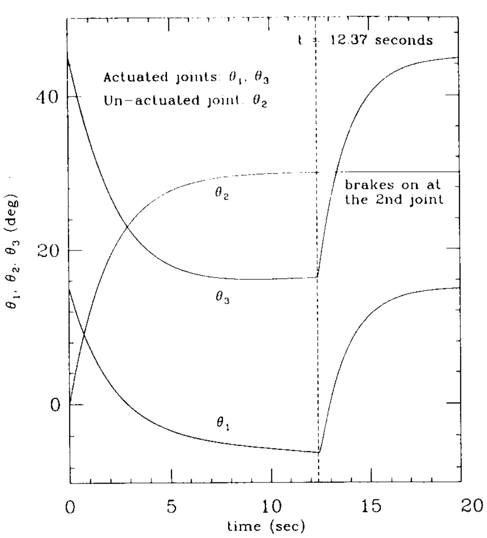

Fig. 5. Trajectories of all the joints of the manipulator for the simulation in Section V-B.

problem of reconfiguring the joints of an underactuated space manipulator system. We analytically proved that if the system maintains zero momentum and if none of the unactuated joints are cyclic coordinates, it will be possible to dissipate the energy of the system and reconfigure the actuated joints in the absence of brakes. We then considered the problem of reconfiguration of all the joints of the manipulator. We showed that if the number of actuated joints are higher than the number of unactuated joints, if there exists sufficient dynamical coupling between the set of actuated and unactuated joints, and if the unactuated joints have brakes, it is possible to control all the joints of the manipulator. Control laws were developed for reconfiguration, and computer simulations were used to verify the efficacy of these control laws. In both the problems that we discussed, we did not control the orientation of the space vehicle. In other words, we controlled the system to equilibrium manifolds.

VIII. Appendix

Lemma 1: A real function $f(t) \in C^2$ defined in (a, b) is concave iff $f(t)'' \le 0, \forall x \in (a,b)$.

Proof: (a) Necessity

Let $x \in (a,b)$. Then for h small enough, $x-h,\ x+h \in (a,b)$. From the definition of concavity [18], $f(x) \ge \frac{1}{2}(f(x-h) + f(x+h))$. Therefore, since $f \in C^2$,

$$f''(x) = \lim_{h\to 0} \frac{f(x-h) + f(x+h) - 2f(x)}{h^2} \le 0. \tag{41}$$

(b) Sufficiency

Let $x, y \in (a,b)$, and $x < y$. For $\lambda \in [0,1]$, and $t = \lambda x + (1-\lambda)y$, the first order Taylor's series approximation of $f(x)$ and $f(y)$ are respectively

$$\begin{aligned} f(x) &= f(t) + f'(t)(x-t) \\ &\quad + f''(\xi_1)(x-t)^2, \quad \xi_1 \in [x,t] \end{aligned} \tag{42}$$

$$\begin{aligned} f(y) &= f(t) + f'(t)(y-t) \\ &\quad + f''(\xi_2)(y-t)^2, \quad \xi_2 \in [t,y]. \end{aligned} \tag{43}$$

It follows that

$$\begin{aligned} \lambda f(x) + (1-\lambda)f(y) &= f(t) + \lambda f''(\xi_1)(x-t)^2 \\ &\quad + (1-\lambda)f''(\xi_2)(y-t)^2 \\ &\le f(t) \qquad \text{since } f''(\xi_1) \le 0, \\ &\quad f''(\xi_2) \le 0. \end{aligned} \tag{44}$$

Therefore the function is concave by definition.

Lemma 2: Let $f(t)$ be a nonpositive function such that $f(t_0) = 0$ and $f(t) < 0$ for some values of t. If the function $f(t)$ is analytic, then $f(t)$ is concave in some open neighborhood of t_0.

Proof: Since the function $f(t)$ is analytic, all derivatives of the function exist and the function can be expanded using Taylor's series as

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t_0)}{n!}(t-t_0)^n. \tag{45}$$

Let us next assume that our function $f(t)$ is not concave in any open neighbothoof of t_0. This implies from Lemma 1 that the condition $f''(t) \le 0$ does not hold good in any open neighborhood of t_0. Therefore eiether $f''(t) \ge 0$, or $f''(t)$ changes sign in every open neighborhood of t_0. If $f''(t) \ge 0$ in every open neighborhood of t_0 then we can show from the corollary of Lemma 1 that $f(t)$ is convex evrywhere. This is not true because $f(t)$ is nonpositive and has a maximum value at $t = t_0$. The other possibility is that $f''(t)$ changes sign in every open neighborhood of t_0. Then $f^{(n)}(t)$ for $n = 2, 3, \cdots, \infty$ changes sign in every open neighborhood of t_0. This implies that $f^{(n)}(t_0) = 0$ for $n = 2, 3, \cdots, \infty$. Additionally, since $f(t)$ is nonpositive and $f(t_0) = 0, f(t)$ achieves a local maximum at t_0. Therefore $f'(t_0) = 0$. Substuting these results in (45), we have $f(t) = 0$. This cannot be true because $f(t)$ is strictly negative for some values of t. We have therefore proved by contradiction that $f(t)$ is concave in some open neighborhood of t_0.

Proof of the Theorem: Part (a) of the theorem can be proven very easily with the help of Lemmas 1 and 2.

To prove part (b), we first realize that $\boldsymbol{x} = 0$ is stable by standard argument since V is locally positive definite and $\dot{V} \leq 0$.

Next, since V is bounded from below by zero and V is nonincreasing $(\dot{V} \leq 0), V \to \alpha, \alpha \geq 0$ as $t \to \infty$.

Since V is analytic and therefore smooth, $\dot{V}$ is uniformly continuous. Hence when $V \to \alpha, \dot{V} \to 0$ as $t \to \infty$, by Barbalat's lemma [19]. Barbalat's lemma [19].

Since V is locally positive definite, $V \to 0 \Rightarrow \boldsymbol{x} \to 0$ as $t \to \infty$. Therefore if we can show that $\alpha = 0$ we can conclude asymptotic stability. We prove $\alpha = 0$ by contradiction. Since $V \to \alpha \neq 0$ and V is locally positive definite, $\exists$ an open neighborhood N of $\boldsymbol{x} = 0$ such that the trajectory of $\boldsymbol{x}(t)$ lies outside $N \, \forall t > T$, and for some $T \geq 0$.

Let $Q = \{\boldsymbol{x} : \dot{V}(\boldsymbol{x}) = 0\}$. Since $\boldsymbol{x}(t)$ converges to Q but lies outside N for large t, the set $W = Q - N$ is nonempty and is the limit set for $\boldsymbol{x}(t)$. If the conditions given by (20) hold then

$$\begin{aligned} V^{(i)}(\boldsymbol{x}) = 0 \qquad \forall i = 1, 2, \cdots, 2k, \qquad \forall \boldsymbol{x} \in W \\ \max_{x \in W} V^{(2k+1)}(\boldsymbol{x}) = -\gamma < 0. \quad (46) \end{aligned}$$

Pick an ϵ arbitrarily small. Since V is analytic and therefore all its derivatives are continuouss, $\exists$ an open neighborhood U of W whose closure U^C does not contain $\boldsymbol{x} = 0$ and $\forall \boldsymbol{x} \in U^C$

$$\begin{aligned} |V^{(i)}(x)| \leq \epsilon, \qquad \forall i = 1, 2, \cdots, 2k \\ V^{(2k+1)}(\boldsymbol{x}) \leq -\gamma + \epsilon. \quad (47) \end{aligned}$$

Since $\boldsymbol{x}(T) \to W$ as $t \to \infty, \exists \, T_1$ such that $x(t) \in U^C \; \forall t \geq T_1$. Now integrating $V^{(2k+1)}(t)$ with respect to time to get V, we have

$$\begin{aligned} V(t) - V(T_1) &= \underbrace{\int_{T_1}^{t} \cdots \int_{T_1}^{t}}_{(2k+1)} V^{(2k+1)}(t) \; dt \\ &\leq \underbrace{\int_{T_1}^{t} \cdots \int_{T_1}^{t}}_{(2k+1)} -(\gamma - \epsilon) \; dt \\ &= -(\gamma - \epsilon) \frac{(t - T_1)^{2k+1}}{(2k+1)!} \\ &\quad + \epsilon \frac{(t - T_1)^{2k}}{(2k)!} + \epsilon \frac{(t - T_1)^{2k}}{(2k-1)!} \\ &\quad + \cdots + \epsilon \, (t - T1) \\ &\triangleq \delta(t). \end{aligned} \quad (48)$$

Hence $V(t) \leq V(T_1) + \delta(t)$, where $\delta(t) \to -\infty$ as $t \to \infty$, since $\gamma > 0$ and ϵ is arbitrarily small. Since $V(T_1) \leq V(t = 0)$ and is therefore bounded, $V(t) \to -\infty$ as $t \to \infty$. This contradicts the fact that $V \geq 0$. Hence $\alpha = 0$ and that implies that the equilibrium point is asymptotically stable.

If all the derivatives of V zero simultaneously, i.e., $V^{(j)}(\boldsymbol{x}) = 0, \forall j = 1, 2, \cdots, \infty$ then the set S is obviously an invariant set. Therefore if S contains only the trivial trajectory, the equilibrium point of the autonomous system will be asymptotically stable from LaSalle's theorem.

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Ranjan Mukherjee (S'89–M'91) was born in Calcutta, India, on August 19, 1965. He received the B.Tech (Honors) degree in Mechanical Engineering from the Indian Institute of Technology, Kharagpur, in 1987, and the M.S. and Ph.D. degrees in 1989 and 1991 respectively, both in Mechanical Engineering from the University of California, Santa Barbara.

In 1991, Dr. Mukherjee joined the faculty of the Mechanical Engineering Department of the Naval Postgraduate School, where he is currently an Assistant Professor. His current research interests include, nonholonomic motion planning, underactuated robot manipulators, space manipulators, and flexible manipulators for surgical operations.

Degang Chen (S'90–M'93) was born in Hubei Province, China, on June 5, 1963. He received the B.S. degree in 1984 in Electrical Engineering from Tsinghua University, Beijing, China, and his M.S. and Ph.D. degrees in 1988 and 1992, respectively, both in electrical engineering, from the University of California, Santa Barbara.

From 1984 to 1986, he was with the Beijing Institute of Control Engineering, a space industry P/D institute. From March 1992 to August 1992, he was the John R. Pierce Instructor of Electrical Engineering at California Institute of Technology. After that, he joined the Iowa State University where he is currently an Assistant Professor. His current teaching and research interests include control system design, nonlinear control systems, adaptive control, robotics, control of articulated flexible structures, space manipulators, and smart materials.

Dr. Chen is the recipient of the Best Paper Award at the 1990 IEEE Conference on Decision and Control.