Proceedings of the 33rd
Conference on Decision and Control
Lake Buena Vista, FL - December 1994

TP-11 3:00

Output Tracking Control of Nonlinear Nonminimum Phase Systems

Degang Chen

Dept. of Electrical Engineering and Computer Engineering
Iowa State University, Ames, IA 50011, email djchen@iastate.edu

Abstract

This paper investigates an extremely important and challenging problem in nonlinear control: output tracking control of nonminimum phase nonlinear systems. Not only does the controller provide stable asymptotic tracking, it also ensures that the transient error will be within a prespecified bound. This is achieved by using a novel approach of stable inversion. In this approach, the nonminimum phase system is first stably inverted off-line to obtain desired (and stable) state and input trajectories that map exactly into the desired output trajectory. Then a feedback controller is designed to stabilize the closed-loop system and to ensure robustness to certain types of uncertainties. The procedures developed may be applied to many important engineering problems, such as aircraft control, rocket control, medical equipment, nondestructive evaluation, and so on.

1 Introduction

A system is nonminimum phase (or has unstable zeros in linear case) if a nonlinear state feedback can hold the system output identically zero while the internal dynamics become unstable [1]. Output tracking control of nonminimum phase systems is a highly challenging problem encountered in many practical engineering applications such as rocket control, aircraft control, flexible-link manipulator control, and elsewhere [e.g., 2-3]. The nonminimum phase property has long been recognized to be a major obstacle in many control problems. It is well-known that unstable zeros cannot be moved with state feedback while the poles can be arbitrarily placed (if completely controllable) [e.g., 4]. In the classical causal inversion approach [5] for exact output tracking, nonminimum phase causes the internal state to become unstable while the output is being tracked [6]. The recently developed nonlinear regulation technique [7] ensures internal stability with asymptotic tracking, but it suffers from large transient errors. In most standard adaptive control [e.g., 8] as well as in nonlinear adaptive control [e.g., 9], all algorithms require that the plant be minimum phase. In the recent nonlinear control literature [e.g., 10], nonminimum phase is again a major barrier in feedback linearization and stabilization of nonlinear systems.

This paper presents a new procedure for designing output tracking controllers for nonminimum phase systems. The new controller will achieve the salient feature of nonlinear regulation: stable asymptotic output tracking, as well as that of classic inversion: high precision without transient errors. This is achieved by using a novel approach of stable inversion. In this approach, the nonminimum phase system is first stably inverted off-line to obtain desired (and stable) state and input trajectories that map exactly into the desired output trajectory. With this, the nonminimum phase control problem is converted into a minimum phase one. Then a feedback controller is designed to stabilize the closed-loop system and to ensure robustness to certain types of uncertainties.

A closely related area of study is the classic inversion theory that was first studied by Brockett and Mesarovic [11]. In Silverman's easy-to-follow step-by-step procedure [12], an input functions defined on $[0, \infty)$ is obtained by solving an initial condition problem for a given output function. Such inverses are necessarily causal but unstable for nonminimum phase systems. These linear inversion results were extended by Hirschorn [5] to real analytic nonlinear systems. Singh ([13] for example) had similar results on nonlinear inversion with modified conditions and considered their applications. Similar to the linear case, these inversion algorithms produce causal inverses for a given desired output $y_d(t)$ and a fixed initial condition $x(t_0)$, leading to unbounded $u(t)$ and $x(t)$ for nonminimum phase systems. This difficulty has been noted for a long time. Singh and Schy [6] have applied these inversion techniques to the control of flexible manipulators. Simulation and experimental results verify that, although exact output tracking can be achieved transiently, internal vibration builds up.

Also closely related is the nonlinear output regulation recently developed by Isidori and Byrnes [7]. This theory provides asymptotic output tracking for a class of nonlinear systems with guaranteed internal stability. The solution of the nonlinear regulator involves solving a set of nonlinear PDE's. Also remain to be tackled is that transient errors can not be controlled precisely and are usually large for nonminimum phase systems. This is verified by its application to flexible manipulators control [14]. This general phenomenon is a fundamental limitation of the regulation approach.

The rest of the paper is organized as follows. The next section sets the basic framework and defines the problem to be solved. In section 3 the stable inversion problem is defined and the solution together with its properties is presented for a class of systems. Section 4 studies the controller design problem. Four different solutions are given, each based on different assumptions and each achieving stable output tracking with prescribed accuracy. Finally, some possible application areas are briefly discussed in the conclusion.

2 Basic Setting

Consider a nonlinear dynamical system described by

$$\Sigma: \begin{cases} \dot{x} = f(x) + g(x)u + \Delta f(x, z) + \Delta g(x, z)u \\ \dot{z} = \varphi(x, z) \\ y = h(x) + \Delta h(x, z) \end{cases}$$

where x is the state vector of the nominal system and is defined on an open neighborhood X of the origin of R^n, z is the state vector of the unmodeled dynamics and is defined on an open neighborhood Z of the origin of $\mathbf{R}^{n_0}$, u is the m-dimensional input vector, y is the p-dimensional output vector, $f(x)$, $g(x)$, $h(x)$, $\Delta f(x, z)$, $\Delta g(x, z)$, $\varphi(x, z)$, $\Delta h(x, z)$ are smooth functions of their arguments with $h(0)=0$ and $f(0)=0$. When $m<p$, it is generally not possible to track all degrees-of-freedom in y. When $m>p$, some of the degrees-of-freedom in u can be used to change the zero dynamics. This project will concentrate on the case $m=p$. In particular, this proposal will assume $m=p=1$ for simplicity. However, all the discussions apply to the case when $m=p>1$ with little changes. When $\Delta f = 0$, $\Delta g = 0$, $\Delta h = 0$, and $z(t) \equiv 0$ with $n_0 = 0$, the reduced system becomes the nominal system:

$$\Sigma_0: \quad \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

A1: Σ_0 is stabilizable and zero input observable.

The class of reference trajectories considered reflects practical considerations. For example, all practical signals have a finite horizon, or are defined over a finite interval of time. In trajectory planning, the reference signals are usually defined by interpolating pre-calculated points. In such cases, it is not practical to try to use an exosystem to generate the reference signal. Hence, the following assumption is made:

A2: The reference output trajectory $y_d(t)$ is a sufficiently smooth function of time satisfying $y_d(t) \equiv 0 \quad \forall t \le t_0$ *and* $\forall t \ge t_f$ where $t_f > t_0$ are finite.

Here, "sufficiently smooth" means that the signal has continuous derivative of any required order. This assumption covers a large family of practical references. Furthermore, it can be easily extended to cover signals whose certain derivatives have a finite horizon.

Let $x(t)$ and $y(t)$ be the state and output trajectories, respectively, of a dynamical system. For a given reference $y_d(t)$ satisfying assumption A2, we define the following tracking properties.

Definition 1: The system is said to achieve

1) Exact-Tracking, if $\|y(t) - y_d(t)\| = 0 \quad \forall t \in R$;
2) ε-Tracking, if $\|y(t) - y_d(t)\| < \varepsilon \quad \forall t \in R$;
3) Asymptotic-Tracing, if $\|y(t) - y_d(t)\| \to 0$ *as* $t \to \infty$;
4) Stable X-Tracking, if X-tracking is achieved with bounded $x(t)$ and $x(t) \to 0$ *as* $t \to \infty$.

Note that "X" can be either "ε", or "exact", or "asymptotic". When applicable, the modifiers can be combined. For example, asymptotic and ε-tracking means that both 2) and 3) are satisfied in definition 1. The causal inversion provides exact-tracking but not stable for nonminimum phase systems; the nonlinear regulation provides stable asymptotic-tracking; the stable inversion is to provide stable exact-tracking; and the proposed controller is to provide stable asymptotic and ε-tracking for the nominal system as well as in the presence of certain uncertainties.

3 Stable Inversion Problem

For the nominal system Σ_0, pose the following:

Problem Statement [15-16]: Given $y_d(t)$ satisfying assumption A2, find bounded $u_d(t)$ and $x_d(t)$ such that

1) $\dot{x}_d(t) = f(x_d(t)) + g(x_d(t))u_d(t) \quad \forall t \in R$ (1)
2) $h(x_d(t)) = y_d(t) \quad \forall t \in R$ (2)
3) $u_d(t) \to 0, \quad x_d(t) \to 0 \quad$ *as* $t \to \pm\infty$

If this problem has a unique solution, system Σ_0 is said to be stable-invertible. This problem has been solved for a class of systems satisfying the following conditions.

A3: System Σ_0 has well-defined relative degree, i.e., $\exists$ $0<r<n$, such that in an open neighborhood of 0, $L_g L_f^i h(x) = 0 \ \forall i = 1, 2, \cdots, r-2$, $L_g L_f^{r-1} h(x) \neq 0$.

Under this assumption, system Σ_0 can be represented in the following normal form

$$\dot{\xi}_i = \xi_{i+1} \qquad i = 1, 2, \cdots, r-1$$

$$\dot{\xi}_r = L_f^r h(x) + L_g L_f^{r-1} h(x) u = \alpha(\xi, \eta) + \beta(\xi, \eta) u \quad (3)$$

$$\dot{\eta} = p(\xi, \eta) \quad (4)$$

Here, $\xi = (\xi_1, \xi_2, \cdots, \xi_r)^T$, $\xi_i = L_f^{i-1} h(x)$, $i = 1, 2, \cdots, r$. The coordinate transformation $(\xi, \eta)^T = \Psi(x)$ is a diffeomorphism with $\Psi(0) = 0$. If u is selected to make

$\dot{\xi}_r \equiv 0$, then $\xi \equiv 0$ and $y = \xi_1 = 0$. The resulting internal dynamics, $\dot{\eta} = p(0, \eta)$, are called the zero dynamics. $\xi = 0$ defines an invariant manifold: the zero dynamics manifold.

A4: The zero dynamics have a hyperbolic fixed point $\xi = 0$, i. e., none of the eigenvalues of the Jacobian matrix $\partial p / \partial \eta(0, 0)$ are on the imaginary axis.

Under assumption A4, there exists an invariant submanifold of the zero dynamics manifold, denoted W^s [17]. W^s has dimension equal to the number of stable eigenvalues of $\partial p / \partial \eta(0, 0)$, and is called the stable manifold. Similarly, there exists an unstable manifold W^u of dimension equal to the number of unstable eigenvalues of $\partial p / \partial \eta(0, 0)$. W^s and W^u are smooth manifolds tangent to the stable and unstable eigenspaces, respectively. W^s and W^u are transversal to each other at $\eta = 0$. Locally W^s is defined by an equation $B^u(\eta) = 0$, and W^u by $B^s(\eta) = 0$.

From eq. (3), since $\beta = L_g L_f^{r-1} h$ is nonsingular,

$$u \triangleq [\beta(\xi, \eta]^{-1} \left[y_d^{(r)} - \alpha(\xi, \eta)\right] \tag{5}$$

is well defined and leads to

$$\dot{\xi}_r = y^{(r)} = y_d^{(r)},$$

$$\xi = \xi_d \triangleq (y_d, \dot{y}_d, \cdots, y_d^{(r_1-1)}).$$

Equation (4) becomes the *reference dynamics* (the zero dynamics driven by the reference output)

$$\dot{\eta} = p(\xi_d, \eta). \tag{6}$$

Theorem 1 [15, 16]: Under assumptions A2-A4, the stable inversion problem for system Σ_0 has a solution if and only if the following two-point boundary value problem has a solution:

$$\dot{\eta} = p(\xi_d, \eta), \tag{6}$$

$$\text{subject to} \quad \begin{cases} B^s(\eta(t_0)) = 0, \\ B^u(\eta(t_f)) = 0. \end{cases} \tag{7}$$

Theorem 2 [18, 19]: Under assumptions A2-A4, the two-point boundary value problem of equations (6-7) has a unique solution provided $\|\xi_d\|_\infty \triangleq \sup_{t\in[t_0, t_f]} \|\xi_d(t)\|_2$ is not too large.

Corollary: Under assumptions A2-A4, system Σ_0 is locally stable-invertible.

Let $\eta_d(t)$ denote the solution to the two-point boundary value problem . Then

$$x_d(t) = \Psi^{-1}(\xi_d(t), \eta_d(t)),$$

$$u_d(t) = [\beta(\xi_d(t), \eta_d(t))]^{-1}[y_d^{(r)} - \alpha(\xi_d(t), \eta_d(t))] \tag{8}$$

define the unique solution to the stable inversion problem. Since ξ_d, η_d have continuous derivatives and Ψ, α, β are smooth functions, it is easy to see the following:

Proposition 1: Under assumptions A2-A4, $x_d(t)$ and $u_d(t)$ are continuously differentiable. Furthermore, if $y_d(t)$ is smooth, $x_d(t)$ and $u_d(t)$ are also smooth.

By condition (7) and by dynamical system theory, the following is immediate:

Theorem 3 [20]: There exist $\gamma_1, \gamma_2 > 0$, $m_1, m_2 > 1$ such that for $t \geq t_f$: $x_d(t) \in W^s$, $\|x_d(t)\| \leq m_1 e^{-\gamma_1(t-t_f)} \|x_d(t_f)\|$, and for $t \leq t_0$: $x_d(t) \in W^u$, $\|x_d(t)\| \leq m_2 e^{+\gamma_2(t-t_0)} \|x_d(t_0)\|$.

Hence $x_d(t) \to 0$ exponentially as $t \to \pm\infty$. The same is true for u_d, since u_d is smooth in x_d. If system Σ_0 is truly nonminimum phase, i. e., dimension of $W^u \neq 0$, then $x_d(t) \neq 0$ and $u_d(t) \neq 0$ for $t \leq t_0$. Hence the inverse is noncausal. If system Σ_0 is actually minimum phase, $W^u = \{0\}$. Then $x_d(t) = 0$, $u_d(t) = 0$, for $t \leq t_0$, and the inverse is causal.

Based on Theorem 1, an algorithm [19] has been developed to solve the stable inversion problem by iteratively linearizing the reference dynamics and decoupling the stable and unstable dynamics. This algorithm has been successfully tested for the flexible manipulator tracking problem [21-22]. In another algorithm [20], the forward dynamics Σ_0 are iteratively discretized to form a set of algebraic (difference) equations. Then the generalized inverse is solved to yield the discretized trajectory for $u_d(t)$. Testing results were also successful. This algorithm is valid because of the following property of the stable inverse.

Theorem 4 [20]: Under assumptions A2-A4, there are infinitely many $u_d(t)$'s that enable system Σ_0 to achieve exact-tracking, out of all $u_*(t)$'s, $u_d(t)$ is the unique one with minimum energy.

4 Output Tracking Control

Once the stable inversion problem is solved, a pair $x_d(t)$, $u_d(t)$, is obtained which by definition solves the exact-tracking problem for the nominal system. However, since the system is nonminimum phase, $u_d(t)$ will be nonzero for all t. Hence, in order to achieve exact tracking, $u_d(t)$ has to be applied starting at $t = -\infty$, which is impossible. Furthermore, since the forward system is not necessarily stable, any small perturbation may lead to divergence from $x_d(t)$, i. e., the loss of output tracking. Therefore, to be practically implementable, $u_d(t)$ with its left tail truncated is used as feed forward and a feedback controller is used to stabilized the closed-loop system.

a) Full State Available for Feedback

First, the possibility of using linear feedback control is studied.

Assumption A5: The nominal system Σ_0 is locally controllable, i. e., $(\partial f / \partial x(0),\ g(0))$ is a controllable.

Let $A = \partial f / \partial x(0)$, $B = g(0)$. Under assumption A5, standard linear techniques can be used to find K such that $A - BK$ is a stable matrix with its eigenvalues placed at desired places. Define a feedback control law:

$$u(t) = \begin{cases} u_d(t) + K(x_d(t) - x(t)) & t - t_0 \geq -T \\ 0 & t - t_0 < -T \end{cases} \quad (9)$$

Then the closed-loop system is:

$$\begin{cases} \dot{x} = f(x) + g(x)u_d + g(x)K(x_d - x) & x(t_0 - T) = 0 \\ \dot{x}_d = f(x_d) + g(x_d)u_d & x_d(-\infty) = 0 \end{cases}$$

Let $x_e(t) = x_d(t) - x(t)$, then

$$\dot{x}_e = f(x_d) - f(x) + (g(x_d) - g(x))u_d - g(x)K(x_d - x)$$

Some algebraic manipulation leads to

$$\dot{x}_e = (A - BK + \Delta(x_d))x_e + O(\|x_e\|^2)$$

where $\Delta(x_d)$ is continuous in x_d with $\Delta(0) = 0$, $O(\|x_e\|^2)$ represents a combined term that is of order of $\|x_e\|^2$. Then it is easy to show that when ξ_d is not too large or x_d is not too large, the x_e dynamics are exponentially stable. Hence there exist $m \geq 1$, $\gamma > 0$, such that

$$\|x_e(t)\| \leq me^{-\gamma(t-t_0+T)}\|x_e(t_0 - T)\|$$

But

$$\|x_e(t_0 - T)\| = \|x_d(t_0 - T)\| \leq m_2 e^{-\gamma_2 T}$$

Therefore

$$\begin{aligned} \|x_e(t)\| &\leq mm_2 e^{-\gamma(t-t_0+T)}e^{-\gamma_2 T} & t \geq t_0 - T \\ &\leq mm_2 e^{-\gamma_2 T} & t \geq t_0 - T \end{aligned}$$

$$\|x_e(t)\| = \|x_d(t)\| \leq m_2 e^{-\gamma_2(t_0 - t)} \leq m_2 e^{-\gamma_2 T} \quad \forall t \leq t_0 - T$$

Since $h(x)$ is smooth, it is locally Lipschitz. Hence,

$$\begin{aligned} \|y_e(t)\| &= \|h(x_d(t)) - h(x(t))\| \\ &\leq L_h\|x_d(t) - x(t)\| = L_h\|x_e(t)\| \\ &\leq L_h mm_2 e^{-\gamma_2 T} \qquad \forall t \in R \end{aligned}$$

Therefore, choosing $T = \dfrac{1}{\gamma_2}\ln\dfrac{L_h mm_2}{\varepsilon}$ guarantees

$$\|y(t) - y_d(t)\| \leq \varepsilon \qquad \forall t \in R.$$

Furthermore, since $\|x_e(t)\| \to 0$ as $t \to \infty$, $\|y(t) - y_d(t)\| \to 0$ as $t \to \infty$ also. Therefore:

Proposition 2: Under assumption A2-A5, the control law (9) leads to stable asymptotic and ε-tracking.

Assumption A5 is a very strong condition. Many nonlinear systems are not locally controllable, but are still stabilizable by using state feedback. One class of systems are characterized by the following.

A 6 : $span\{g,\ ad_f g,\ \cdots,\ ad_f^{n-2}g\}$ is involutive with constant rank, and $span\{g,\ ad_f g,\ \cdots,\ ad_f^{n-1}g\}$ has rank n, in a neighborhood of $0 \in \mathbf{R}^n$.

Assumption A6 is the necessary and sufficient condition for the nominal system to be feedback linearizable. Under this assumption, there exists a smooth function $\phi(x)$ such that the relative degree between $\phi(x)$ and u is n. Let $\varsigma_i = L_f^{i-1}\phi(x)$, $\varsigma_{di} = L_{f(x_d)}^{i-1}\phi(x_d), i = 1, \cdots, n$. Then the diffeomorphism $\varsigma = [\varsigma_1,\ \varsigma_2,\ \cdots,\ \varsigma_n]^T = \varsigma(x)$ forms a change of coordinates. The nominal system in the new coordinates has the following normal form representation:

$$\begin{cases} \dot{\varsigma}_i = \varsigma_{i+1} & i = 1,\ \cdots,\ n-1 \\ \dot{\varsigma}_i = L_f^n\phi(x) + L_g L_f^{n-1}\phi(x)u \end{cases}$$

This suggests the following feedback control law for $t \geq t_0 - T$ ($u(t) = 0$ for $t < t_0 - T$):

$$\begin{aligned} u(t) = {} & [L_g L_f^{n-1}\phi(x)]^{-1}[-L_f^n\phi(x) + L_{f(x_d)}^n\phi(x_d) \\ & + a_{n-1}(\varsigma_{dn} - \varsigma_n) + \cdots + a_1(\varsigma_{d2} - \varsigma_2) + a_0(\varsigma_{d1} - \varsigma_1)] \end{aligned} \quad (10)$$

where $s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ is Hurwitz. In the original coordinates, the control law is

$$\begin{aligned} u(t) = {} & [L_g L_f^{n-1}\phi(x)]^{-1}[-L_f^n\phi(x) + L_{f(x_d)}^n\phi(x_d) \\ & + a_{n-1}(L_{f(x_d)}^{n-1}\phi(x_d) - L_f^{n-1}\phi(x)) + \cdots \\ & + a_1(L_{f(x_d)}\phi(x_d) - L_f\phi(x)) + a_0(\phi(x_d) - \phi(x))] \end{aligned}$$

Let $e_i = \varsigma_{di} - \varsigma_i$, $e = [e_1,\ e_2,\ \cdots,\ e_n]^T$, then

$$\dot{e} = \begin{bmatrix} 0 & & I & \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} e \qquad t \geq t_0 - T$$

Therefore $e \to 0$ exponentially, i.e., $\varsigma \to \varsigma_d$ as $t \to \infty$. Since $\varsigma = \varsigma(x)$, $x(t) \to x_d(t)$ exponentially as $t \to \infty$. Hence for $t \geq t_0 - T$

$$\|x(t) - x_d(t)\| \leq me^{-\gamma(t-t_0+T)}\|x(t_0 - T) - x_d(t_0 - T)\|$$

Following similar argument as before, one obtains:

Proposition 3: Under assumptions A2, A3, and A6, the control law (10) leads to stable asymptotic and ε-tracking for proper T.

b) Selected Output Feedback

In many applications, the full state is not available for feedback, or it has too many components to measure, or it leads to a controller that is too complicated. Then some kind of output feedback is preferred. Since the regulated output y leads to a difficult nonminimum phase control problem, a different set of output variables are selected for feedback. For example, in flexible-link manipulator control, the tip trajectory is the regulated output to be tracked and joint torque is the input. But this output renders the input/output mapping nonminimum phase. For a given tip trajectory, the stable inversion problem can be solved first to yield a desired joint angle trajectory. Then the joint angle can be used for feedback to stabilize the closed-loop system. This task is relatively easier since it is a minimum phase system control problem.

Let $\zeta = s(x)$ be measurement for feedback. Then $\zeta_d = s(x_d)$ will be the desired trajectory for ζ to track. The choice of ζ needs to satisfy two requirements:

A7: The measurement ζ can be chosen such that there is well defined relative degree and the resulting zero dynamics are exponentially stable.

Under this assumption, the following normal form is another representation of the nominal system

$$\begin{cases} \zeta^{(r_1)} = \alpha_1(x) + \beta_1(x)u \\ \dot{\eta}_1 = p_1(x) = p_1(\xi_1, \eta_1) \end{cases}$$

where $\xi_1 = (\zeta, \dot{\zeta}, \cdots, \zeta^{(r_1-1)})^T$ and r_1 is the relative degree. Let $\xi_{d1} = (\zeta_d, \dot{\zeta}_d, \cdots, \zeta_d^{(r_1-1)})^T$. Then the desired trajectory for η_1 satisfies

$$\dot{\eta}_{1d} = p_1(\xi_{1d}, \eta_{1d}).$$

And ζ_d satisfies

$$\zeta_d^{(r_1)} = \alpha_1(x_d) + \beta_1(x_d)u_d,$$

$$u_d = [\beta_1(x_d)]^{-1}[\zeta_d^{(r_1)} - \alpha_1(x_d)]. \quad (11)$$

Let $\zeta_e = \zeta_d - \zeta$, then we can have

$$u = [\beta_1(x)]^{-1}[-\alpha_1(x) + \zeta_d^{(r_1)} + a_{r_1-1}\zeta_e^{(r_1-1)} + \cdots + a_0\zeta_e]$$

$$\zeta_e^{(r_1)} + a_{r_1-1}\zeta_e^{(r_1-1)} + \cdots + a_1\dot{\zeta}_e + a_0\zeta_e = 0.$$

Clearly, if $s^{r_1} + a_{r_1-1}s^{r_1-1} + \cdots + a_1 s + a_0$ is Hurwitz, $\zeta_e \to 0$ exponentially. However, the control law as defined above is not realizable, since x is not available for feedback. Noting that

$$u = \beta_1^{-1}[\zeta_d^{(r_1)} - \alpha_1] + \beta_1^{-1}[a_{r_1-1}\zeta_e^{(r_1-1)} + \cdots + a_1\dot{\zeta}_e + a_0\zeta_e]$$

and comparing equation (11) suggest the following control law for $t \ge t_0 - T$

$$u = u_d + [\beta_1(x_d)]^{-1}[\frac{a_{r_1-1}}{\tau}\zeta_e^{(r_1-1)} + \cdots + \frac{a_0}{\tau^{r_1}}\zeta_e] \quad (12)$$

where $0 < \tau < 1$ is a small parameter used to scale the eigenvalues. Substituting this control law leads to

$$\zeta^{(r_1)} = \alpha_1(x) + \beta_1(x)u_d + \beta_1(x)[\beta_1(x_d)]^{-1} \bullet [\frac{a_{r_1-1}}{\tau}\zeta_e^{(r_1-1)} + \cdots + \frac{a_1}{\tau^{r_1-1}}\dot{\zeta}_e + \frac{a_0}{\tau^{r_1}}\zeta_e]$$

Let $\Delta\alpha = \alpha_1(x_d) - \alpha_1(x)$, $\Delta\beta = \beta_1(x_d) - \beta_1(x)$. Rearranging terms leads to

$$\zeta_e^{(r_1)} + (1 + \Delta\beta[\beta_1(x_d)]^{-1})[\frac{a_{r_1-1}}{\tau}\zeta_e^{(r_1-1)} + \cdots + \frac{a_0}{\tau^{r_1}}\zeta_e] = \Delta\alpha + \Delta\beta u_d$$

Since $\Delta\alpha$, $\Delta\beta$ depend on η_1 also, the stability of the above equation has to be studied together with

$$\dot{\eta}_{1e} = \dot{\eta}_{1d} - \dot{\eta}_1 = p_1(\xi_{1d}, \eta_{1d}) - p_1(\xi_1, \eta_1).$$

Clearly, if $\Delta\alpha = 0$, $\Delta\beta = 0$, $\zeta_e \to 0$ exponentially, hence $\xi_{1e} \to 0$, $\eta_{1e} \to 0$ exponentially. Next note that $\|\Delta\alpha\| \le L_\alpha(\|x_d\|)\left\|\begin{pmatrix}\xi_{1e}\\ \eta_{1e}\end{pmatrix}\right\|$, $\|\Delta\beta\| \le L_\beta(\|x_d\|)\left\|\begin{pmatrix}\xi_{1e}\\ \eta_{1e}\end{pmatrix}\right\|$, where $L_\alpha(\|x_d\|)$ and $L_\beta(\|x_d\|)$ are Lipschitz constants depending on $\|x_d\|$ with $L_\alpha(0) = 0$, $L_\beta(0) = 0$. Therefore, by regularity of exponential stability, $\xi_{1e} \to 0$, $\eta_{1e} \to 0$ exponentially when $\|x_d\|$ is small enough. This in turn through a diffeomorphism implies $x \to x_d$ exponentially. The allowable size of x_d can be made large by regulating τ. Once exponential stability is established, similar procedures as before can be applied to establish ε-tracking of the regulated output. To summarize, we have

Proposition 4: Under assumptions A2-A4 and A7, the control law (12) based on measured output feedback renders system Σ_0 to achieve stable asymptotic and ε-tracking.

c) Observer Based State Feedback

Another alternative to deal with unavailability of full state is to use state observers. Then the following assumption is in order.

A8: Σ_0 is locally observable.

The full state feedback control law (9) is changed to

$$u(t) = u_d(t) + K(x_d(t) - \hat{x}(t)) \quad (13)$$

where $\hat{x}$ is the estimate of the state vector. The observer is of the form:

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + L(h(x) - h(\hat{x})) \quad (11)$$

Let $\tilde{x}(t) = \hat{x}(t) - x(t)$, then

$$\dot{\tilde{x}} = f(\hat{x}) - f(x) + (g(\hat{x}) - g(x))u + L(h(x) - h(\hat{x}))$$

After some manipulation, this is of the form

$$\dot{\tilde{x}} = (A - LC + \Delta_o(x_d))\tilde{x} + O\left(\left\|\begin{matrix}x_e\\ \tilde{x}\end{matrix}\right\|^2\right)$$

In the linear case, the stability of the observer can be established independent of that of the plant. However, in the nonlinear case, due to the nonlinear coupling, this is no longer the situation. Let $x_e = x_d - x$, then

$$\dot{x}_e = f(x_d) + g(x_d)u_d - f(x) - g(x)u_d - g(x)K(x_d - \hat{x})$$
$$= f(x_d) - f(x) + (g(x_d) - g(x))u_d - g(x)Kx_e + g(x)K\tilde{x}$$

This can be put into the form:

$$\dot{x}_e = (A - BK + \Delta_p(x_d))x_e + (BK + \Delta_c(x_d))\tilde{x} + O\left(\left\|\begin{matrix}x_e\\ \tilde{x}\end{matrix}\right\|^2\right)$$

Here, $\Delta_i(x_d)$, $i = o, p, c$, are continuous in x_d and satisfy $\Delta_i(0) = 0$. Furthermore, proper choice of K, L ensures that $A - BK$ and $A - LC$ are stable matrices. Hence, $\tilde{x}$, $x_e \to 0$ exponentially when $\Delta_i = 0$, $i = o, p, c$. By continuity, $\tilde{x}$, $x_e \to 0$ exponentially if x_d is not too large. Again the allowable size of x_d can be enlarged to some extent by proper choice of K and L. Similar procedure can be used to establish ε-tracking.

Proposition 5: Under assumptions A2-A5 and A8, the observer based control laws (13) and (11) can render the system Σ_0 to achieve stable asymptotic and ε-tracking.

5 Conclusion

This paper has presented procedures for designing output tracking controllers that apply to nonminimum phase as well as minimum phase systems. Not only does the controller provide stable asymptotic tracking of a desired reference trajectory, it also ensures that transient errors will be within a prespecified bound. The robustness in stability and tracking performance to certain perturbations further adds to the great potential of the new approach to many important engineering applications. For example, it can be used to the aircraft altitude control that is known to be a nonminimum phase control problem. By essentially removing transient errors, better riding comfort can be achieved. Another application example is the rocket tracking problem which is again a well-known nonminimum phase problem. The high precision tracking performance provided by the new approach should lead to greatly improved accuracy of hits. This, in turn, means large increase of impact power due to the cubic rule, i.e., increasing the hit accuracy by 10 is equivalent to increasing the head power by 1000. The techniques and algorithms developed in the stable inversion theory may also be applied to other inverse problems in engineering.

References

1 A. Isidori, *Nonlinear Control Systems: An Introduction*, Springer-Verlag, New York, 1989.

2 B. Paden, D. Chen, R. Ledesma, and E. Bayo, "Exponentially Stable Tracking Control for Multi-Link Flexible Manipulators," *ASME J. Dynamics, Measurement and Control*, vol. 115, no. 1, pp. 53-59, 1993.

3 R.A. Nichols, R.T. Reichert, and W.J. Rugh, "Gain Scheduling for H-Infinity Controllers: A Flight Control Example," *IEEE Trans. On Control Systems Technology*, vol. 1, no. 2, pp. 69-79, 1993.

4 W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed., Spinger-Verlag, New York, 1985.

5 R.M. Hirschorn, "Invertibility of Multivariable Nonlinear Control Systems," *IEEE Trans. On Automatic Control*, vol. 24, no. 6, pp. 855-865, 1979.

6 S.N. Singh and A.A. Schy, "Control of Elastic Robotic Systems by Nonlinear Inversion and Modal Damping," *J. Dynamic Systems Measurement and Control*, vol. 108, 1986.

7 A. Isidori and C.I. Byrnes, "Output Regulation of Nonlinear Systems," *IEEE Trans. on Automatic Control*, vol. 35, no. 2, pp. 131-140, 1990.

8 K.S. Narendra and A.M. Annaswamy, *Stable Adaptive Systems*, Prentice Hall, Englewood Cliffs, 1989.

9 R. Ghanadan and G.L. Blankenship, "Adaptive Approximate Tracking and Regulation of Nonlinear Systems," *Proceedings of the 32nd IEEE Conference on Decision and Control*, pp. 2654-2659, San Antonio, December, 1993.

10 N.A. Mahmoud and H.K. Khalil, "Asymptotic Stabilization of Minimum Phase Nonlinear Systems Using Output Feedback," *Proceedings of the 32nd IEEE Conference on Decision and Control*, pp. 1960-1965, San Antonio, December, 1993.

11 R.W. Brockett and M.D. Mesarovic, "The Reproducibility of Multivariable Systems," *J. Mathematical Analysis and Applications*, vol. 11, pp. 548-563, 1965.

12 L.M. Silverman, "Inversion of Multivariable Linear Systems," *IEEE Trans. On Automatic Control*, vol. 14, no. 3, pp. 270-276, 1969.

13 S.N. Singh, "A Modified Algorithm for Invertibility in Nonlinear Systems," *IEEE Trans. On Automatic Control*, vol. 26, no. 2, pp. 595-599, 1981.

14 P. Lucibello and M.D. Di Benedetto, "Output Tracking for a Nonlinear Flexible Arm," *submitted to J. of ASME*.

15 D. Chen, *Nonlinear adaptive Control of Electro-Mechanical Systems*, PhD Dissertation, University of California, Santa Barbara, 1992.

16 D. Chen and B. Paden, "Stable Inversion of Nonlinear Nonminimum Phase Systems," *Proceedings of Japan/USA Symposium on Flexible Automation*, pp. 791-797, San Francisco, July, 1992.

17 S. Wiggins, *Introduction to Applied Nonlinear Dynamical System and Chaos*, Springer-Verlag, New York, 1990.

18 B. Paden and D. Chen, "State Space Conditions for the Invertibility of Nonminimum Phase Systems," *Proceedings of the ASME Winter Annual Meeting*, pp. 37-41, Anaheim, CA, Nov. 1992.

19 D. Chen, "An Iterative Solution to Stable Inversion of Nonlinear Nonminimum Phase Systems," *Proceedings of the 1993 American Control Conference*, pp. 2960-2964, San Francisco, June, 1993.

20 H. Zhao and D. Chen, "A Minimum Energy Approach to Stable Inversion of Nonlinear Nonminimum Phase Systems," to be presented at *the 1994 American Control Conference*.

21 H. Zhao and D. Chen, "Stable Inversion and Its Application to Tip Trajectory Tracking for Flexible Link Manipulators," *Proceedings of ASME International Conference on Inverse Problems in Engineering*, Palm Coast, Florida, June, 1993.

22 H. Zhao and D. Chen, "Exact and Stable Tip Trajectory Tracking for Multilink Flexible Manipulators," *Proceedings of the 32nd IEEE Conference on Decision and Control*, pp. 1371-1376, San Antonio, December, 1993.