Stable Inverison for Exact and Stable Tracking Controllers: A Flight Control Example*

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Abstract

Output tracking for missile flight control is challenging due to the nonminimum phase problem. Among existing methods for output tracking, the regulation approach usually leads to large transient errors and the dynamic inversion approach results in unbounded internal dynamics for nonminimum phase systems. By using a stable inversion approach, this paper presents a new missile autopilot controller that achieves an exact and stable output tracking without any transient or steady state errors and internal vibration.

1 Introduction

The problem of output tracking control for nonlinear systems is of great importance from both the theoretical and the practical viewpoints, and it has been studied extensively for many years. There are two basic approaches to attack this problem. Using feedback regulation is one way which involves stabilizing the closed-loop system so as to achieve asymptotic tracking of a class of reference inputs. The second approach is to implement a tracking controller with a feed-forward signal generated by an inverse system coupled with a stabilizing feedback scheme.

For linear multivariable cases, the asymptotic tracking problem was solved by [1, 2] and subsequently crystallized as the internal model principle [3]. The matrix equations defining the asymptotic tracking controllers for linear systems were translated to nonlinear partial differential equations in the nonlinear cases [4]. Although nonlinear partial differential equations are only numerically tractable for systems of low orders, solutions for tracking periodic trajectories have been developed based on Fourier series [5, 6]. However, the transient error phenomenon is still a fundamental limitation of the regulation approach.

The transient behavior can be precisely controlled by using stabilizing feedback together with a feed-forward signal generated by an inverse system. For linear multivariable systems the inversion problem has been solved to a large degree by Brockett and Mesarovic [7] and Silverman [8]. However, these inverses are all causal. The linear inversion results were extended to nonlinear real-analytic systems and conditions for the invertibility of these systems have been derived by Hirschorn [9, 10]. Singh [11], besides, had similar results on nonlinear inversion with some modified conditions. All these inversion algorithms produce causal inverses for a given desired output $y_d(t)$ and a fixed initial condition $x(t_0)$. Unbounded control input $u(t)$ and state trajectory $x(t)$ were produced for nonminimum phase systems. This fundamental difficulty has been noted for a long time.

Motivated by the success of the noncausal inverse dynamics approach [12], the notion of stable inversion has recently been developed [13] in an effort to find feedforward signals for the tracking controller. The problem has been solved for a class of nonlinear nonminimum phase systems with a well-defined relative degree and hyperbolic zero dynamics. A numerical procedure has also been developed [14] for constructing stable inverses based on iterative linearization and decomposition of the stable/unstable subspaces. This approach to output tracking avoids difficulties in both regulation and classical inversion while preserves the advantages of both, and is applied to achieve an exact and stable output trajectory tracking in a missile autopilot example.

2 Stable Inversion

Consider the multivariable nonlinear control systems of the form

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

$$y = h(x), \tag{2}$$

where system state x is defined on an open neighborhood X of the origin of $\mathbf{R}^n$ and input $u \in \mathbf{R}^m$ and output $y \in \mathbf{R}^m$. The mappings $f(x)$ and $g_i(x)$ (the ith column of $g(x)$), for $i = 1, 2, \ldots, m$, are smooth vector fields defined on X, and $h_i(x)$, for $i = 1, 2, \ldots, m$, are smooth functions on X, and $f(0) = 0$ and $h(0) = 0$. For such systems, the stable inversion problem has been stated by

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Chen [13] as follows:

Stable Inversion Problem: *Given a smooth reference output trajectory $y_d(t)$ with compact support, find a control input $u_d(t)$ and a state trajectory $x_d(t)$ such that*
1) $u_d(t)$ and $x_d(t)$ satisfy the differential equation

$$\dot{x}_d(t) = f(x_d(t)) + g(x_d(t))u_d(t);$$

2) exact output tracking is achieved

$$h(x_d(t)) = y_d(t);$$

3) $u_d(t)$ and $x_d(t)$ are bounded and

$$u_d(t) \to 0, \quad x_d(t) \to 0 \quad \text{as} \quad t \to \pm\infty.$$

We call $x_d(t)$ the desired state trajectory and $u_d(t)$ the nominal control input. These can be incorporated into a dead-beat controller which achieves the exact output trajectory tracking by using the nominal control input as a feed-forward signal and $x - x_d$, the error signal, as a feedback signal to the controller. That is mainly the idea of controllers design in this paper.

With $\mathbf{N} \overset{def}{=} \{1, 2, \ldots\}$, $r \overset{def}{=} (r_1, r_2, \ldots, r_m)^T \in \mathbf{N}^m$ and $y : \mathbf{R}^n \to \mathbf{R}^m$, we define

$$|r| \overset{def}{=} \sum_{i=1}^{m} r_i,$$

$$y^{(r)} \overset{def}{=} (\frac{d^{r_1}y_1}{dt^{r_1}}, \frac{d^{r_2}y_2}{dt^{r_2}}, \ldots, \frac{d^{r_m}y_m}{dt^{r_m}})^T,$$

$$L_f^r y \overset{def}{=} (L_f^{r_1}y_1, L_f^{r_2}y_2, \ldots, L_f^{r_m}y_m)^T,$$

$$L_g^{\mathbf{1}} y \overset{def}{=} (L_{g_1}y, L_{g_2}y, \ldots, L_{g_m}y).$$

Assume that the system has a well-defined relative degree $r \in \mathbf{N}^m$ at the equilibrium point at the origin. To partially linearize the system, we define $\xi_k^i \overset{def}{=} y_i^{(k-1)}$ for all $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, r_i$, and denote

$$\begin{aligned} \xi &\overset{def}{=} (\xi_1^1, \xi_2^1, \ldots, \xi_{r_1}^1, \xi_1^2, \ldots\ldots, \xi_{r_m}^m)^T \\ &\overset{def}{=} (y_1, \dot{y}_1, \ldots, y_1^{(r_1-1)}, y_2, \ldots\ldots, y_m^{(r_m-1)})^T. \end{aligned} \quad (3)$$

Choose η, an $n - |r|$ dimensional function on $\mathbf{R}^n$ such that

$$(\xi^T, \eta^T)^T = \Phi(x) = (\phi_1(x), \ldots, \phi_n(x))^T$$

forms a change of coordinates with $\Phi(0) = 0$ [15]. In this new coordinate system, the system equations (1)–(2) becomes

$$\left\{ \begin{array}{ll} \dot{\xi}_1^i = \xi_2^i & \\ \vdots & \text{for } i = 1, \ldots, m \\ \dot{\xi}_{r_i-1}^i = \xi_{r_i}^i & \\ \dot{\xi}_{r_i}^i = \alpha_i(\xi, \eta) + \beta_i(\xi, \eta)u & \\ \dot{\eta} = q_1(\xi, \eta) + q_2(\xi, \eta)u, & \end{array} \right. \quad (4)$$

which, in a more compact form, is equivalent to

$$y^{(r)} = \alpha(\xi, \eta) + \beta(\xi, \eta)u, \quad (5)$$

$$\dot{\eta} = q_1(\xi, \eta) + q_2(\xi, \eta)u, \quad (6)$$

where

$$\alpha(\xi, \eta) = L_f^r h(\Phi^{-1}(\xi, \eta)),$$

$$\beta(\xi, \eta) = L_g^{\mathbf{1}} L_f^{r-1} h(\Phi^{-1}(\xi, \eta)),$$

$$q_{1_i}(\xi, \eta) = L_f(\phi_i(\Phi^{-1}(\xi, \eta))),$$

$$q_{2_i}(\xi, \eta) = L_g^{\mathbf{1}}(\phi_i(\Phi^{-1}(\xi, \eta))),$$

for all $|r| + 1 \leq i \leq n$. Note that $\alpha(0,0) = 0$ and $q_{1_i}(0,0) = 0$ since $f(0) = 0$. Define the following feedback control law

$$u \overset{def}{=} [\beta(\xi, \eta)]^{-1}(\nu - \alpha(\xi, \eta)), \quad (7)$$

and choose $\nu = y_d^{(r)}$, immediately we have

$$\xi = \xi_d \overset{def}{=} (y_{d_1}, \dot{y}_{d_1}, \ldots, y_{d_1}^{(r_1-1)}, y_{d_2}, \ldots\ldots, y_{d_m}^{(r_m-1)})^T,$$

and equation (6) becomes the so-called reference dynamics,

$$\dot{\eta} = p(y_d^{(r)}, \xi_d, \eta), \qquad \eta \in \mathbf{R}^{n-|r|} \quad (8)$$

where

$$p(y_d^{(r)}, \xi_d, \eta) \overset{def}{=}$$
$$q_1(\xi_d, \eta) + q_2(\xi_d, \eta)[\beta(\xi_d, \eta)]^{-1}(y_d^{(r)} - \alpha(\xi_d, \eta)).$$

It is now clear that an integration of the reference dynamics gives rise to a trajectory of the original states through the inverse coordinate transformation $x = \Phi^{-1}(\xi, \eta)$ and an input trajectory by equation (7). The problem is how to integrate the reference dynamics to generate a bounded state trajectory and a bounded input which would solve the stable inversion problem since the reference dynamics may be unstable in both positive and negative time directions in general.

For reference trajectories with compact support, the reference dynamics becomes autonomous zero dynamics for t outside the compact interval $[t_0, t_f]$. Assume that $\eta = 0$ is a hyperbolic equilibrium point of the zero dynamics. It has been shown that the stable inversion problem is equivalent to the following two-point boundary value problem [13]:

$$\dot{\eta} = p(y_d^{(r)}, \xi_d, \eta), \quad (9)$$

subject to

$$\left\{ \begin{array}{l} B^s(\eta(t_0)) = 0 \\ B^u(\eta(t_f)) = 0, \end{array} \right. \quad (10)$$

where $B^s(\eta) = 0$ characterizes the unstable manifold denoted as W^u, and $B^u(\eta) = 0$ the stable manifold W^s. It has also been shown that this two-point boundary value problem locally has a unique solution [14]. Once the equivalent two-point boundary value problem is solved, the desired state trajectory $x_d(t)$ and the nominal control input $u_d(t)$ can be constructed as follows:

$$x_d = \Phi^{-1}(\xi_d, \eta), \tag{11}$$

$$u_d = [\beta(\xi_d, \eta)]^{-1}(y_d^{(r)} - \alpha(\xi_d, \eta)), \tag{12}$$

where $\beta(\cdot) = L_g^1 L_f^{r-1} h(\Phi^{-1}(\cdot))$, and $\alpha(\cdot) = L_f^r h(\Phi^{-1}(\cdot))$.

It is clear that any state trajectories and control inputs satisfying the reference dynamics (9) would produce the exact output tracking of the reference trajectory. However, it is a nontrivial numerical problem to solve the equivalent two-point boundary value problem because of the instability of the reference dynamics in both positive and negative time directions.

3 An Iterative Solution

In this section an iterative linearization approach to the solution of the two-point boundary value problem which was presented by Chen [14] is described. In each iteration, the differential equation (9) and its boundary conditions (10) are linearized along the solution obtained from the previous step to form a new linear time-varying two point boundary value problem. This linear problem is then solved involving integrations in two directions and the solution is taken to be this step's new approximation. The iteration continues until some convergence criterion is met.

To initialize, we take $\eta^0(t) = 0$ for all t. Let η^{k-1} be the solution obtained in the $(k-1)$th step. Let η^k denote the new corrected solution to be solved in the kth step, the current step. Linearizing the right hand sides of the differential equation (9) and the boundary conditions (10) along η^{k-1} and setting η to η^k, we have

$$\dot{\eta}^k = \frac{\partial p}{\partial \eta}(\bar{\xi}_d, \eta^{k-1})(\eta^k - \eta^{k-1}) + p(\bar{\xi}_d, \eta^{k-1}), \tag{13}$$

subject to

$$\frac{\partial B^s}{\partial \eta}(\eta^{k-1}(t_0))(\eta^k(t_0) - \eta^{k-1}(t_0)) + B^s(\eta^{k-1}(t_0)) = 0,$$

$$\frac{\partial B^u}{\partial \eta}(\eta^{k-1}(t_f))(\eta^k(t_f) - \eta^{k-1}(t_f)) + B^u(\eta^{k-1}(t_f)) = 0,$$

where

$$\bar{\xi}_d \stackrel{def}{=} (y_d^{(r)}, \xi_d).$$

By defining the following symbols,

$$A^k(t) \stackrel{def}{=} \frac{\partial p}{\partial \eta}(\bar{\xi}_d(t), \eta^{k-1}(t)),$$

$$b^k(t) \stackrel{def}{=} -\frac{\partial p}{\partial \eta}(\bar{\xi}_d(t), \eta^{k-1}(t))\eta^{k-1}(t) + p(\bar{\xi}_d(t), \eta^{k-1}(t)),$$

$$C_s^k \stackrel{def}{=} \frac{\partial B^s}{\partial \eta}(\eta^{k-1}(t_0)),$$

$$C_u^k \stackrel{def}{=} \frac{\partial B^u}{\partial \eta}(\eta^{k-1}(t_f)),$$

$$\mu^k \stackrel{def}{=} \frac{\partial B^s}{\partial \eta}(\eta^{k-1}(t_0))\eta^{k-1}(t_0) - B^s(\eta^{k-1}(t_0)),$$

$$\nu^k \stackrel{def}{=} \frac{\partial B^u}{\partial \eta}(\eta^{k-1}(t_f))\eta^{k-1}(t_f) - B^u(\eta^{k-1}(t_f)),$$

we can rewrite equation (13) and its boundary conditions in the following format,

$$\dot{\eta}^k = A^k(t)\eta^k + b^k(t), \tag{14}$$

subject to

$$C_s^k \eta^k(t_0) = \mu^k, \quad C_u^k \eta^k(t_f) = \nu^k. \tag{15}$$

That is a linear time-varying two-point boundary value problem we have to solve in each step.

The idea of solving the linear time-varying two-point boundary value problem is to try to separate the stable and unstable dynamics and to integrate the stable part forward in time and the unstable part backward in time. A technique from linear-quadratic optimal control is used and for convenience, we drop the superscript k in the followings.

To decouple the stable and unstable dynamics, we apply a change of coordinates

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} C_s \\ C_u \end{bmatrix} \eta.$$

Since $B^s(\eta) = 0$ is the condition for η to be on the unstable manifold, therefore $B^s(\eta)$ may be viewed as the stable part of η. Hence, in the linear approximation, z_1 is, roughly speaking, picking up the stable part of η, and similarly, z_2 the unstable part.

Let $T = [\, T_s \; T_u \,]$ be the inverse transformation matrix, then

$$\eta = [\, T_s \quad T_u \,] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{16}$$

In this new coordinates,

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} C_s \\ C_u \end{bmatrix} \dot{\eta} = \begin{bmatrix} C_s \\ C_u \end{bmatrix} (A(t)\eta + b(t)).$$

Substituting the inverse transformation (16) into the above equation, we can arrange it into

$$\dot{z}_1 = A_{11}(t)z_1 + A_{12}(t)z_2 + b_1(t), \tag{17}$$

$$\dot{z}_2 = A_{21}(t)z_1 + A_{22}(t)z_2 + b_2(t), \tag{18}$$

with initial and final conditions specified respectively as $z_1(t_0) = \mu$ and $z_2(t_f) = \nu$, and the matrices are defined as follows:

$$A_{11}(t) \stackrel{def}{=} C_s A(t) T_s, \quad A_{12}(t) \stackrel{def}{=} C_s A(t) T_u,$$

$$A_{21}(t) \stackrel{def}{=} C_u A(t) T_s, \quad A_{22}(t) \stackrel{def}{=} C_u A(t) T_u,$$

$$b_1(t) \stackrel{def}{=} C_s b(t), \quad b_2(t) \stackrel{def}{=} C_u b(t).$$

Since z_1 and z_2 satisfy a pair of linear coupled differential equations, it is easy to see that the solutions are also linearly related. Therefore, there exist a vector function $g(t)$ and a matrix function $S(t)$ of suitable dimensions, such that

$$z_2(t) = S(t)z_1(t) + g(t), \tag{19}$$

with suitable final value conditions

$$S(t_f) = 0 \quad \text{and} \quad g(t_f) = \nu. \tag{20}$$

Differentiating both sides of equation (19) yields

$$\dot{z}_2(t) = \dot{S}(t)z_1(t) + S(t)\dot{z}_1(t) + \dot{g}(t).$$

Substituting the values of $\dot{z}_1$ and $\dot{z}_2$ from (17) and (18) and comparing the coefficients of $z_1(t)$ lead to

$$\dot{S} = A_{21} + A_{22}S - SA_{11} - SA_{12}S, \tag{21}$$

$$\dot{g} = (A_{22} - SA_{12})g + (b_2 - Sb_1), \tag{22}$$

with final conditions specified in equation (20).

Since equation (21) contains only known functions except S, it can be integrated backward in time to get $S(t)$. Once this is done, equation (22) can also be integrated backward in time to solve for $g(t)$. With $S(t)$ and $g(t)$ as known functions, equation (17), which can be rewritten as

$$\dot{z_1} = (A_{11}(t) + A_{12}(t)S(t))z_1 + b_1(t) + A_{12}(t)g(t), \tag{23}$$

can be integrated forward in time to obtain $z_1(t)$. Finally, the algebraic equation (19) is used to obtain $z_2(t)$. Thus, the kth iterative step is finished by transforming $[z_1^T, z_2^T]^T$ back into η by inverse transformation (16).

Note that by taking the limits on both sides of equations (13) and its boundary conditions, which are the linearized form in the kth step of the equivalent two-point boundary value problem, we immediately obtain the original equivalent two-point boundary value problem. It means that if the numerical iteration converges, the limit η solves the nonlinear two-point boundary value problem. Thus solving the stable inversion problem.

4 Missile Autopilot Example

A closed-loop controller for missile autopilot is designed in this section using stable inversion. The controller uses the nominal control as its feed-forward input which is superimposed by a stabilizing feedback signal. A stable and exact output trajectory tracking is expected.

Consider the longitudinal rigid-body dynamics of a missile traveling at Mach 3 at an altitude of 20,000 ft. The pitch-axis model involving angle of attack and pitch rates is described as follows

$$\begin{aligned} \dot{\alpha} &= K_\alpha M f_1(\alpha)\cos\alpha + K_\alpha M d_n \delta \cos\alpha + q && (24)\\ \dot{q} &= K_q M^2 f_2(\alpha) + K_q M^2 d_m \delta, && (25)\end{aligned}$$

where the aerodynamic coefficients are given by

$$f_1(\alpha) = a_n\alpha^3 + b_n\alpha|\alpha| + c_n\alpha \tag{26}$$

$$f_2(\alpha) = a_m\alpha^3 + b_m\alpha|\alpha| + c_m\alpha, \tag{27}$$

and α is the angle of attack in degrees; q pitch rate in degrees per second; δ actual tail deflection angle in degrees, and M the Mach number. The tail fin actuator dynamics is approximated by a first-order lag

$$\dot{\delta} = -\frac{1}{\tau}\delta + \frac{1}{\tau}u, \tag{28}$$

where u represents commanded tail fin deflection angle in degrees, and the output is normal acceleration

$$y = K_z M^2 f_1(\alpha) + K_z M^2 d_n \delta. \tag{29}$$

The missile with parameters listed in Table 1 is utilized as the physical model here [16]. The two key assumptions to ensure the equivalence between the stable inversion problem and the two-point boundary value problem are verified as follows.

$K_\alpha = (0.7)P_o S/mv_s$	$K_q = (0.7)P_o Sd/I_y$
$K_z = (0.7)P_o S/qm$	$P_o = 973.3\, lbs/ft^2$
$S = 0.44\, ft^2$	$m = 13.98\, slugs$
$v_s = 1036.4\, ft/s$	$d = 0.75\, ft$
$I_y = 182.5\, slug \cdot ft^2$	$\tau = 1/15$
$a_n = 0.000103\, deg^{-3}$	$a_m = 0.000215\, deg^{-3}$
$b_n = -0.00945\, deg^{-2}$	$b_m = -0.0195\, deg^{-2}$
$c_n = -0.1696\, deg^{-1}$	$c_m = 0.051\, deg^{-1}$
$d_n = -0.034\, deg^{-1}$	$d_m = -0.206\, deg^{-1}$

Table 1: Details of Pitch-Axis Missle Model

The relative degree of the system is unit which may be easily verified by the fact that input u is explicitly involved in the first derivative of the output:

$$\dot{y} = K_z M^2 \frac{df_1(\alpha)}{d\alpha}\dot{\alpha} + K_z M^2 d_n(-\frac{1}{\tau}\delta + \frac{1}{\tau}u), \tag{30}$$

where

$$K_z M^2 d_n \frac{1}{\tau} = \beta \neq 0.$$

By change of variables

$$\begin{cases} \alpha = \eta_1 \\ q = \eta_2 \\ \delta = (K_z M^2 d_n)^{-1}(\xi - k_z M^2 f_1(\eta_1)), \end{cases} \tag{31}$$

and let $\xi = y = y_d$, the desired output, the reference dynamics is given by

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 + K_\alpha (K_z M)^{-1} y_d \cos\eta_1 \\ \dot{\eta}_2 &= K_q M^2 f_2 - K_q M^2 d_m d_n^{-1} f_1 + K_q d_m (K_z d_n)^{-1} y_d. \end{aligned}$$

When output is identically zero corresponding to the reference trajectory outside the compact support, the Jacobian matrix corresponding to the zero dynamics, reference dynamics with reference trajectory being identically zero, is given by

$$J = \begin{bmatrix} 0 & 1 \\ K_q M^2(c_m - d_m d_n^{-1} c_n) & 0 \end{bmatrix}. \tag{32}$$

The equilibrium point at the origin of the zero dynamics is hyperbolic since it contains one dimensional stable manifold and one dimensional unstable one, which may be seen from the eigenvalues of the Jacobian with the fact that $c_m > 0, d_m < 0, d_n < 0$, and $c_n < 0$.

To implement our stable inversion method to the design of tracking controllers, the equivalent two-point boundary value problem is derived as follows. The linearized version is directly obtained in favor of the numerical iterations.

The iterative linearization at each step k along the solutions obtained in the previous step $k-1$ of the reference dynamics is given by

$$\begin{bmatrix} \dot{\eta}_1^k \\ \dot{\eta}_2^k \end{bmatrix} = A(t) \begin{bmatrix} \eta_1^k \\ \eta_2^k \end{bmatrix} + b(t), \qquad (33)$$

where

$$A(t) = \begin{bmatrix} -K_\alpha(K_z M)^{-1} y_d \sin\eta_1 & 1 \\ k_q M^2(f_2'(\eta_1) - d_m d_n^{-1} f_1'(\eta_1)) & 0 \end{bmatrix},$$

$$b(t) = \begin{bmatrix} K_\alpha(K_z M)^{-1} y_d(\cos\eta_1 + \eta_1 \sin\eta_1) \\ K_q M^2(f_2 - d_m d_n^{-1} f_1 - f_2' \eta_1 + \\ + d_m d_n^{-1} f_1' \eta_1) + K_q d_m (K_z d_n)^{-1} y_d \end{bmatrix},$$

with the notation

$$f_i' \stackrel{def}{=} \frac{df_i(\eta_1)}{d\eta_1},$$

and η_i is evaluted at η_i^{k-1}.

Looking for the boundary conditions of the equivalent two-point boundary value problem, we form the matrix X_s by taking as columns the eigenvectors and the generalized eigenvectors of $A(t)$ at some fixed time t corresponding to eigenvalues having negative real parts, and X_u, those corresponding to eigenvalues having positive real parts. Then, we have

$$A(t) \begin{bmatrix} X_s & X_u \end{bmatrix} = \begin{bmatrix} X_s & X_u \end{bmatrix} \begin{bmatrix} J_s & 0 \\ 0 & J_u \end{bmatrix}, \qquad (34)$$

where J_s and J_u are the corresponding Jordan forms. Denote

$$\begin{bmatrix} Y_s \\ Y_u \end{bmatrix} = \begin{bmatrix} X_s & X_u \end{bmatrix}^{-1}.$$

From (34) we obtain

$$Y_s A(t) X_u = 0 \quad and \quad Y_u A(t) X_s = 0. \qquad (35)$$

Since we know that $\eta(t)$ belongs to the unstable manifold for all $t \le t_0$, and it belongs to the stable manifold for all $t \ge t_f$, therefore $\eta(t_0)$ (respectively $\eta(t_f)$) can be written as a linear combination of the columns of X_u (respectively X_s):

$$\eta(t_0) = X_u Z_u \quad and \quad \eta(t_f) = X_s Z_s. \qquad (36)$$

Combining equations (35) and (36), we have

$$Y_s A(t_0)\eta(t_0) = Y_s A(t_0) X_u Z_u = 0,$$

$$Y_u A(t_f)\eta(t_f) = Y_u A(t_f) X_s Z_s = 0.$$

Denoting $C_s = Y_s A(t_0)$ and $C_u = Y_u A(t_f)$, we obtain the linear time-varying two-point boundary value problem for each iteration

$$\dot{\eta}(t) = A(t)\eta(t) + b(t) \qquad (37)$$

subject to

$$C_s \eta(t_0) = 0 \quad and \quad C_u \eta(t_f) = 0. \qquad (38)$$

Assume that the reference output profile has been chosen as shown in Figure 1. The missile is required to track the desired trajectory in its normal acceleration. The controller is imposed by the following structure

$$u = u_d - k(x - x_d), \qquad (39)$$

where x denotes the state variables of the forward dynamics, $x = (\alpha, q, \delta)^T$. Feedback gian, k, is chosen such as to stabilize the forward dynamics linearized at the origin. In this example,

$$k = \begin{bmatrix} -51.1878 & -2.9860 & 7.6265 \end{bmatrix},$$

which puts the poles of the linearized dynamics at -6.0016, 5.3984 and -15.0000 into -50, $-40 + i20$ and $-40 - i20$. The nominal control input $u_d(t)$ and the desired state trajectory $x_d(t)$ are obtained through the following iterative step:

- Step 1: Set $\eta^0(t) = 0$ for all t.
- Step 2: Linearize the reference dynamics along $\eta^0(t)$ to get (17)–(23).
- Step 3: Integrate equation (21) backward in time to get $S(t)$.
- Step 4: Integrate equation (22) backward in time to get $g(t)$.
- Step 5: Integrate equation (23) forward in time to get $z_1(t)$ and get $z_2(t)$ by (19).
- Step 6: Compute $\eta(t) = \begin{bmatrix} C_s \\ C_u \end{bmatrix}^{-1} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$.
- Step 7: If $\|\eta - \eta^0\|$ is greater than a given threshold, then $\eta^0 = \eta$ and go to step 2, else continue to step 8.
- Step 8: Compute desired state $x_d(t)$ by inverse coordinates transformation $x_d = \Phi^{-1}(\xi_d, \eta)$ and nominal input $u_d(t)$ by $u_d = [\beta(\xi_d, \eta)]^{-1}(y_d^{(r)} - \alpha(\xi_d, \eta))$, which is

$$u_d = (K_z M^2 d_n)^{-1} \tau(\dot{y}_d + (\tau^{-1} - K_\alpha M f' \cos\eta_1) y_d - \\ - K_z M^2 (f' \eta_2 + \tau^{-1} f_1)).$$

Simulation results are shown in Figure 1. It can be seen from the results that an exact output tracking is achieved, and the internal dynamics of the system is stable with desired state trajectories and the nominal control input approach zero as time goes to either plus or minus infinity.

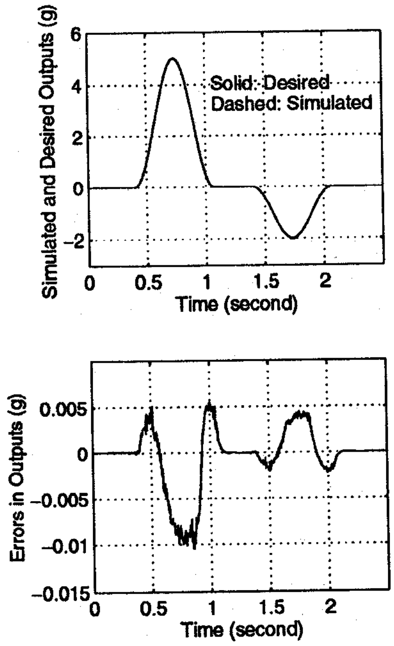

Figure 1: Simulation Results by Stable Inversion.

5 Conclusion

The stable inversion approach to the design of output tracking controllers for nonlinear nonminimum phase systems is successfully applied to the trajectory tracking of a missile autopilot example. The key assumptions on well-defined relative degree and hyperbolicity of the fixed point of the zero dynamics are satisfied. Simulation results demonstrate that the stable inversion approach is very effective for obtaining exact output tracking for this flight control example. This approach is expected to perform equivalently well for other realistic nonminimum phase systems. Future work will be on efficient numerical algorithms for constructing stable inverses and on new applications of stable inversion.

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