Nonlinear Inversion-Based Output Tracking

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Abstract—**An inversion procedure is introduced for nonlinear systems which constructs a bounded input trajectory in the preimage of a desired output trajectory. In the case of minimum phase systems, the trajectory produced agrees with that generated by Hirschorn's inverse dynamic system; however, the preimage trajectory is noncausal (rather than unstable) in the nonminimum phase case. In addition, the analysis leads to a simple geometric connection between the unstable manifold of the system zero dynamics and noncausality in the nonminimum phase case. With the addition of stabilizing feedback to the preimage trajectory, asymptotically exact output tracking is achieved. Tracking is demonstrated with a numerical example and compared to the well-known Byrnes–Isidori regulator. Rather than solving a partial differential equation to construct a regulator, the inverse is calculated using a Picard-like interaction. When preactuation is not possible, noncausal inverse trajectories can be truncated resulting in the tracking-error transients found in other control schemes.**

I. Introduction

Tracking control and regulation are common problems in applications and have thus attracted considerable attention from control researchers. Asymptotic tracking has been solved for a given reference trajectory in the context of linear-quadratic optimal control in [1]. Also, for linear systems, the asymptotic regulation and tracking of signals generated by finite-dimensional linear systems has been studied in a general framework by Francis and Wonham [2]. These authors show that the tracking problem is solvable if and only if a set of linear matrix equations is solvable. In the nonlinear case, the Francis–Wonham equations have been generalized to a first-order partial differential algebraic equation (PDE) by Byrnes and Isidori [3]. This fundamental work has been augmented with tests for approximate solvability of the Byrnes–Isidori PDE [4] and methods for optimal regulator design [5]. In addition, extensions to the Byrnes–Isidori regulator have been described in [6] and [7].

In this paper we introduce an inversion-based approach to exact nonlinear output tracking control. Briefly, the control strategy is as follows. Let $y_d(\cdot)$ be a desired output trajectory of a nonlinear system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t)$$

$$y(t) = h(x(t))$$

(defined more precisely later). The idea is to use control of the form $u(t) = u_d(t) + K(x_d(t) - x(t))$, where $(u_d(\cdot), x_d(\cdot))$ is the desired input-state trajectory (found through inversion) satisfying

$$\begin{aligned}\dot{x}_d(t) &= f(x_d(t)) + g(x_d(t))u_d(t)\\ y_d(t) &= h(x_d(t)).\end{aligned}$$

The feedback term, $K(x_d(t) - x(t))$, is chosen to stabilize the system along the desired state trajectory.

The Byrnes–Isidori regulator can be applied to any trajectory generated by a given exosystem, but it requires the nontrivial solution of a set of partial differential algebraic equations. We trade this requirement—to solve the partial differential algebraic equations—for the tracking of a specific trajectory (rather than any one of a family). Moreover, no exosystem is required, and the specification of trajectories is simplified. We do, however, introduce boundedness and integrability requirements on the trajectory. The key to our approach is finding a bounded inverse, even for nonminimum phase nonlinear systems, for use in generating feedforward inputs. In contrast to the inversion approach of Hirschorn [8] where unstable zero dynamics lead to unbounded responses of the inverse system, we introduce a nonlinear operator which is noncausal in the nonminimum phase case. The resulting desired input trajectories are also noncausal, and we use preactuation to establish initial conditions in the nonminimum phase case, in contrast to setting initial conditions as is done in [9]. Other methods that result in approximate tracking can be found in [10] and [11].

Noncausal feedforward can be used in the case where trajectory preview is possible or truncated to a causal signal at the cost of introducing transient tracking errors. Such noncausal character is seen in the linear quadratic setting, but the use of exact inverses in nonlinear tracking control is new. The noncausal inverses used here are a generalization of the work by Bayo [12] in flexible multibodies which have been applied to the control of flexible-link robots in [13]. Recent work by Meyer *et al.* [14] removes some of the restrictions in our theorems, and makes extensions in the context of air-traffic control.

Our paper is organized in the following format. Section II is devoted to the formulation of a nonlinear operator denoted $\mathbb{N}$ and to establishing sufficient conditions for convergence of a constructive iteration. In the next section, we describe the

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application of $\mathbb{N}$ to the development of a nonlinear regulator based on dynamic inversion. In Section IV, the methodology is illustrated with an example, and our conclusions are made in Section V.

II. A Noncausal Nonlinear Operator

Here, we develop a nonlinear operator, $\mathbb{N}$, which is central to the stable inversion of nonlinear nonminimum phase systems. We begin with its linear counterpart, denoted $\boldsymbol{A}$.

A. Linear Operator A

Consider the linear system with boundary conditions at $+\infty$ and $-\infty$

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad x(\pm\infty) = 0 \tag{1}$$

where A has no $j\omega$-axis eigenvalues, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, the matrices A and B have compatible dimensions, and $u \in L_1 \cap L_\infty$. Linear system (1) has a unique solution on $(-\infty, \infty)$ (see [15, ch. 3]). Without loss of generality, we choose state coordinates such that

$$A = \begin{bmatrix} A_- & 0 \\ 0 & A_+ \end{bmatrix}$$

where A_- (A_+), respectively, has all of its eigenvalues in the open left- (right-) half plane. It is straightforward to verify that

$$x(t) = \int_{-\infty}^{\infty} \phi(t-\tau)Bu(\tau)\,d\tau \tag{2}$$

is the unique solution to (1) if

$$\phi(t) = \begin{bmatrix} 1(t)e^{tA_-} & 0 \\ 0 & -1(-t)e^{tA_+} \end{bmatrix} \tag{3}$$

and $1(\cdot)$ is the unit step function. The Green's function (see, e.g., [16]) or impulse response for (1) is $\phi(\cdot)B$. The state transition matrix usually associated with the initial value problem $\dot{x}(t) = Ax(t) + Bu(t)$; $x(0) = 0$ is replaced, in the boundary-value problem (1), by $\phi(\cdot)$. We refer to $\phi(\cdot)$ as the bounded state-transition matrix.

Example: For

$$A = \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix} \tag{4}$$

we have

$$\phi(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{if} \quad t > 0$$

$$\phi(t) = \begin{bmatrix} 0 & 0 \\ 0 & -e^{t} \end{bmatrix} \qquad \text{if} \quad t < 0. \tag{5}$$

□

We define two norms on $\phi(\cdot)$ which will be used later.

Definition 1—Two Norms on Linear State Transition Matrices:

$$\|\phi(\cdot)\|_\alpha \triangleq \sum_j \max_k \|\phi_{j,k}(\cdot)\|_1 \tag{6}$$

$$\|\phi(\cdot)\|_\beta \triangleq n \sup_\tau \max_{j,k} |\phi_{j,k}(\tau)| \tag{7}$$

where $\phi_{j,k}(t)$ is the jth row and kth column element of $\phi(t) \in \mathbb{R}^{n\times n}$. Note that $\|\phi(\cdot)\|_\alpha < \infty$ implies that the system is hyperbolic. However, this condition is not needed for the norm described by (7) to be bounded, as long as the poles on the imaginary axis are simple.

Definition 2—Linear Operator $\boldsymbol{A}$: Define the input-to-state operator $\boldsymbol{A}$: $L_1 \cap L_\infty \to L_1 \cap L_\infty$ by

$$x(t) \triangleq \int_{-\infty}^{\infty} \phi(t-\tau)Bu(\tau)\,d\tau. \tag{8}$$

Note that since both $u(\cdot)$ and $\phi(\cdot)$ are in $L_1 \cap L_\infty$, the right-hand side of (8) can be evaluated using Fourier transform.

B. Nonlinear Operator $\mathbb{N}$

For the system $\dot{x}(t) = f(x(t), u(t))$; $x(\pm\infty) = 0$ we proceed to develop $\mathbb{N}$, the nonlinear analog of $\boldsymbol{A}$. We start with a few definitions which will be used to show that a Picard-like iteration (using the linear operator $\boldsymbol{A}$) provides a contraction that converges to a bounded solution of this nonlinear equation.

Definition 3—Solution Definition: Let $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$, and $f(\cdot,\cdot)$: $\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$. Then for a given $u(\cdot) \in L_1 \cap L_\infty$, $x(\cdot) \in L_1 \cap L_\infty \cap C^0$ is called a solution to the differential equation with boundary conditions

$$\frac{d}{dt}x(t) = f(x(t), u(t)); \qquad x(\pm\infty) = 0 \tag{9}$$

if 1) the above equation is satisfied almost everywhere (a.e.) in time t, where $-\infty < t < \infty$; and 2) $\lim_{t\to\pm\infty} x(t) = 0$. Note that $u(\cdot) \in L_1 \cap L_\infty$ is fairly restrictive since it implies that $u(\cdot) \in L_p$ for all $1 \le p \le \infty$. The operator defined here can be generalized in some cases to $u(\cdot) \in L_\infty$ (see [14] and [17]).

We approximate the nonlinear $f(x(t), u(t))$ by $Ax(t)$, and the following definition expresses that the vector field formed by the error in this approximation is locally Lipshitz in both x and u.

Definition 4—Locally Approximately Linear (l.a.l) Condition: The mapping $f(\cdot,\cdot)$: $\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is defined to be locally approximately linear (l.a.l) in an r neighborhood of $(0, 0)$ with positive real constants (K_1, K_2) if there exist $A \in \mathbb{R}^{n\times n}$ and $r > 0$, such that for $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^n$, and $v(t) \in \mathbb{R}^n$ all with $\|\cdot\|_\infty$ norms less than r, the following local Lipschitz condition holds for all t:

$$\begin{aligned}\|\{f[x(t), u(t)] - Ax(t)\} - \{f[y(t), v(t)] - Ay(t)\}\|_1 \\ \le K_1\|x(t) - y(t)\|_1 + K_2\|u(t) - v(t)\|_1.\end{aligned} \tag{10}$$

This condition on f implies that 1) $f(0, 0) = 0$, 2) $f(\cdot,\cdot)$ is locally Lipshitz, and 3) if $f(\cdot,\cdot)$ is smooth, then A can be chosen as $D_x f(x, u)|_{(0,0)}$. Note that the continuity of $f(\cdot,\cdot)$ in both of its arguments is necessary, but smoothness is not required to satisfy the l.a.l. condition. For example, $f(x, u) = x + 0.1|x| + u$ is l.a.l., but any $f(x, u)$ with a step discontinuity in x at the origin is not l.a.l.

Given $f(\cdot, \cdot)$ which is l.a.l, let A be chosen to satisfy (10) and $\phi(\cdot)$ be the bounded state-transition matrix associated with A. This will be assumed for the rest of the paper. The nonlinear operator will be developed through iterations based on a linear approximation of $f(\cdot, \cdot)$ by A. To apply a contraction-mapping, we impose the following condition on $f(\cdot, \cdot)$ and its approximation A.

Definition 5—Condition 1: Given a positive scalar r, we say that the approximation of $f(x, u)$ by Ax satisfies Condition 1 if f: $\mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is l.a.l (Definition 4) in an r neighborhood with Lipshitz constants (K_1, K_2), where K_1 and K_2 satisfy

$$K_1 < \frac{1}{\|\phi(\cdot)\|_\alpha} \tag{11}$$

$$\|u(\cdot)\|_1 K_2 < \frac{r(1 - \|\phi(\cdot)\|_\alpha K_1)}{\|\phi(\cdot)\|_\beta} \tag{12}$$

for all $u(\cdot) \in U \triangleq L_1 \cap L_\infty \cap B_r$, where

$$B_r \triangleq \{\overline{u}(\cdot) | \overline{u}(t) \in \mathbb{R}^p, \text{ and } \|\overline{u}(\cdot)\|_1 + \|\overline{u}(\cdot)\|_\infty < r\}. \tag{13}$$

Inequality (11) implies that $\dot{x}(t) = Ax(t)$ has a hyperbolic equilibrium at $x = 0$ (i.e., $\|\phi(\cdot)\|_\alpha$ is bounded). Also, bounds on K_1 and K_2 restrict the degree to which f differs from the linear map.

C. Existence and Uniqueness Theorem

In the following theorem we define a map from $u(\cdot) \in L_1 \cap L_\infty \cap B_r$ into solutions $x(\cdot)$ of (9). A similar mapping from bounded continuous functions (BC) into BC is studied in [15].

Theorem 1: If f satisfies Condition 1, then for all $u(\cdot) \in L_1 \cap L_\infty \cap B_r$ there exists a unique solution $x(\cdot)$ of $\frac{d}{dt}x(t) = f(x(t), u(y))$; $x(\pm\infty) = 0$.

Remark: The boundary condition $x(-\infty) = 0$ has the practical implication that with a finite look-ahead these solutions can be approximated (in the $L_1 \cap L_\infty$ sense) in the interval $(-T, \infty)$ to a high accuracy, provided T is large enough. The other boundary condition may be relaxed (see, for example, [14] and [15]).

Proof: Construct the sequence $\{x_m(\cdot)\}_{m=0}^{\infty}$ by defining $x_0(\cdot) \equiv 0$, and

$$\begin{aligned} x_{m+1}(t) &= P_u(x_m)(t) \\ &\triangleq \int_{-\infty}^{\infty} \phi(t-\tau)\{f[x_m(\tau), u(\tau)] - Ax_m(\tau)\}\, d\tau \end{aligned} \tag{14}$$

where $P_{u(\cdot)}[x_m(\cdot)]$ is denoted for simplicity by $P_u(x_m)$. We claim that this sequence converges to the required solution of (9). The proof consists of four lemmas which are proven in some generality because a few intermediate results are required later.

Lemma 1:

$$P_u\colon\ L_1 \cap L_\infty \to L_\infty \cap C^0.$$

Proof:

$$\begin{aligned} \|P_u(x)(t) - P_v(y)(t)\|_\infty &\le \|P_u(x)(t) - P_v(y)(t)\|_1 \\ &\triangleq \left\| \int_{-\infty}^{\infty} \phi(t-\tau)B(\tau)\, d\tau \right\|_1 \end{aligned} \tag{15}$$

where

$$B(\tau) \triangleq \{f[x(\tau), u(\tau)] - Ax(\tau)\} - \{f[y(\tau), v(\tau)] - Ay(\tau)\}.$$

Hence, for all t, we have

$$\begin{aligned} &\|P_u(x)(t) - P_v(y)(t)\|_\infty \\ &\le \left\| \int_{-\infty}^{\infty} \phi(t-\tau)B(\tau)\, d\tau \right\|_1 \\ &\triangleq \sum_{j=1}^{n} \int_{-\infty}^{\infty} \left| \sum_{k=1}^{n} (\phi_{j,k}(t-\tau)B_k(\tau)) \right| d\tau \\ &\le \sum_{j=1}^{n}\sum_{k=1}^{n} \int_{-\infty}^{\infty} |\phi_{j,k}(t-\tau)||B_k(\tau)|\, d\tau \\ &\le \left(\sup_{\tau}\max_{j,k} |\phi_{j,k}(\tau)|\right) \sum_{k=1}^{n} \int_{-\infty}^{\infty} |B_k(\tau)|\, d\tau \\ &\triangleq \|\phi(\cdot)\|_\beta \int_{-\infty}^{\infty} \|B(\tau)\|_1\, d\tau \\ &\le \|\phi(\cdot)\|_\beta \int_{-\infty}^{\infty} (K_1\|x(\tau) - y(\tau)\|_1 \\ &\quad + K_2\|u(\tau) - v(\tau)\|_1)\, d\tau \qquad \text{[from (10)]} \\ &\triangleq \|\phi(\cdot)\|_\beta (K_1\|x(\cdot) - y(\cdot)\|_1 + K_2\|u(\cdot) - v(\cdot)\|_1). \end{aligned} \tag{16}$$

Setting $v(\cdot)$ and $y(\cdot)$ to zero establishes that the range of P_u is contained in L_∞ and is uniformly continuous in t. □

This lemma implies that the integrand of iteration (14) belongs to $L_1 \cap L_\infty$ and is uniformly continuous in t.

Lemma 2:

$$P_u\colon\ L_1 \cap L_\infty \to L_1.$$

Proof:

$$\begin{aligned} &\|P_u(x)(\cdot) - P_v(y)(\cdot)\|_1 \\ &\qquad \triangleq \sum_{j=1}^{n} \int_{-\infty}^{\infty} \left| \sum_{k=1}^{n} \int_{-\infty}^{\infty} \phi_{j,k}(t-\tau)B_k(\tau)\, d\tau \right| dt \end{aligned} \tag{17}$$

where $B(\tau) \triangleq \{f[x(\tau), u(\tau)] - Ax(\tau)\} - \{f[y(\tau), v(\tau)] - Ay(\tau)\}$. Hence

$$\begin{aligned} &\|P_u(x)(\cdot) - P_v(y)(\cdot)\|_1 \\ &\le \sum_{j=1}^{n}\sum_{k=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi_{j,k}(t-\tau)B_k(\tau)|\, d\tau\, dt \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi_{j,k}(t-\tau)B_k(\tau)|\, dt\, d\tau \\ &\le \sum_{j}^{n} \left(\max_k \|\phi_{j,k}(\cdot)\|_1\right) \|B(\cdot)\|_1 \\ &\triangleq \|\phi(\cdot)\|_\alpha \|B(\cdot)\|_1 \\ &\le \|\phi(\cdot)\|_\alpha (K_1\|x(\cdot) - y(\cdot)\|_1 + K_2\|u(\cdot) - v(\cdot)\|_1) \end{aligned} \tag{18}$$

using (10) as in the proof of Lemma 1. Setting $v(\cdot) \equiv 0$ and $y(\cdot) \equiv 0$ completes the proof. □

Corollary to Lemma 2: P_u is a contraction on L_1.

Proof: Set $v(\cdot) = u(\cdot)$ and substitute (11) in (18). □

Lemma 3:

$$x_m(\cdot) \in L_1, \text{ and } \|x_m(t)\|_\infty \le r, \qquad \text{for all } m = 1, 2, \cdots.$$

Proof: 1) $x_0(\cdot) \in L_1 \cap L_\infty \cap C^0$, and 2) $x_m(\cdot) \in L_1 \cap L_\infty \cap C^0 \Rightarrow x_{m+1}(\cdot) \in L_1 \cap L_\infty \cap C^0$ (from Lemmas 1 and 2). By induction $x_m(\cdot) \in L_1 \cap L_\infty \cap C^0$ for all $m = 1, 2, \cdots$, *provided* $x_m(\cdot)$ *remains in the neighborhood*, i.e., $\|x_m(\cdot)\|_\infty < r$. To see that this proviso is satisfied, observe that

$$\begin{aligned}\|x_m(\cdot)\|_1 &= \|x_m(\cdot) - x_0(\cdot)\|_1 \le \sum_{k=0}^{m-1} \|x_{k+1}(\cdot) - x_k(\cdot)\|_1 \\ &\le \sum_{k=0}^{m-1} (\|\phi(\cdot)\|_\alpha K_1)^k \|\phi(\cdot)\|_\alpha K_2 \|u(\cdot)\|_1 \\ &\quad \text{[using (18)]} \\ &\le \frac{\|\phi(\cdot)\|_\alpha K_2}{1 - K_1\|\phi(\cdot)\|_\alpha} \|u(\cdot)\|_1 \\ &\quad (\text{since } K_1\|\phi(\cdot)\|_\alpha < 1). \end{aligned} \tag{19}$$

Further, from (16)

$$\|x_m(t)\|_\infty \le \|\phi(\cdot)\|_\beta (K_1\|x_{m-1}(\cdot)\|_1 + K_2\|u(\cdot)\|_1), \quad \forall t \in (-\infty, \infty). \tag{20}$$

Substituting (19) of $\|x_{m-1}(\cdot)\|_1$ in the right-hand side of the above equation, and using (12) (Condition 1), we obtain for all t

$$\|x_m(t)\|_\infty \le \frac{K_2\|\phi(\cdot)\|_\beta \|u(\cdot)\|_1}{1 - K_1\|\phi(\cdot)\|_\alpha} \le r. \tag{21}$$

□

Lemma 4: $\lim_{m\to\infty} x_n(t) \triangleq x(t)$ exists for all t, and $x(\cdot) = P_u(x)(\cdot)$.

Proof: From (16), for all t

$$\begin{aligned} &\|x_m(t) - x_{m-1}(t)\|_\infty \\ &\le K_2\|\phi(\cdot)\|_\beta \|u(\cdot)\|_1 (K_1\|\phi(\cdot)\|_\alpha)^m \\ &\Rightarrow x_m(t) - x_{m-1}(t) \to 0 \text{ as } m \to \infty \\ &\quad (\text{since } K_1\|\phi(\cdot)\|_\alpha < 1). \end{aligned} \tag{22}$$

Hence, there exists $x(t) \triangleq \lim_{m\to\infty} x_m(t)$ for all t. Next we show that this pointwise limit function $x(\cdot)$ is the fixed point of the contraction $x_{n+1}(\cdot) = P_u(x_n)(\cdot)$.

From (22) we have, for all t

$$\lim_{m\to\infty} \|P_u(x_m)(t) - x_m(t)\|_\infty \triangleq \lim_{m\to\infty} \|x_{m+1}(t) - x_m(t)\|_\infty = 0$$

for all t

$$\Rightarrow \lim_{m\to\infty} [P_u(x_m)(t) - x_m(t)] = 0. \tag{23}$$

From (16) we have the Lipshitz condition

$$\|P_u(x)(t) - P_u(y)(t)\|_\infty \le K_1\|\phi\|_\beta \|x(\cdot) - y(\cdot)\|_1. \tag{24}$$

This means that $P_u(x)(t)$ is continuous with respect to $x \in L_1$ in the $\|\cdot\|_\infty$ norm, $\lim_{m\to\infty} P_u(x_m)(t) = P_u(\lim_{m\to\infty} x_m)(t)$, and

$$P_u(x)(t) - x(t) = 0. \tag{25}$$

$x_m(\cdot)$ converges to $x(\cdot)$ uniformly in t, and therefore $x(\cdot)$ is continuous. From the definition of P_u (14) and (25)

$$\begin{aligned} x(t) &= P_u(x)(t) \\ &\triangleq \int_{-\infty}^{\infty} \phi(t-\tau)[f(x(\tau), u(\tau)) - Ax(\tau)]\, d\tau. \end{aligned} \tag{26}$$

This implies that $x(\cdot)$ is a fixed point of the contraction P_u.□

From Lemmas 3 [see (21)] and 4, the uniform pointwise convergence of the uniformly bounded sequence $x_m(\cdot)$ implies that $\|x(t)\|_\infty \le r$ for all t, i.e., the solution is bounded. To see that $x(\cdot)$ satisfies (9), differentiate (26) to obtain[1]

$$\begin{aligned} \frac{d}{dt}x(t) &= \int_{-\infty}^{\infty} \left[\frac{d}{dt}\phi(t-\tau)\right] \{f(x(\tau), u(\tau)) - Ax(\tau)\}\, d\tau \\ &= \int_{-\infty}^{\infty} [A\phi(t-\tau) + I^{n\times n}\delta(t-\tau)] \\ &\quad \times \{f(x(\tau), u(\tau)) - Ax(\tau)\}\, d\tau \qquad \text{[from (3)]} \\ &= f(x(t), u(t)). \end{aligned} \tag{27}$$

By construction, $x_m(\cdot)$ is uniformly continuous for all $m = 1, 2, \cdots$ (Lemma 1). Since $x_m(\cdot) \in L_1$, from (27) the l.a.l condition and Barbalats lemma [18] implies that $x_m(t) \to 0$ as $t \to \pm\infty$. The uniform convergence of the sequence $\{x_m(\cdot)\}_{m=0}^{\infty} \Rightarrow x(t) \to 0$ as $t \to \pm\infty$, and thus the boundary conditions on the solution are satisfied. Also, by the contraction mapping theorem, $x(\cdot)$ is the unique fixed point [satisfies (26)], and hence the unique solution to (9). This concludes the proof of Theorem 1. □

Definition of the Nonlinear Operator $\mathbb{N}$*:* Theorem 1 defines a mapping $\mathbb{N}$ from a given input $u(\cdot) \in U$ [U defined by (13)] to a state trajectory $x(\cdot) \in L_1 \cap L_\infty \cap C^0$ as follows: $\mathbb{N}(u(\cdot)) \triangleq \lim_{m\to\infty} x_{m,u}(\cdot)$, where $x_{m,u}(\cdot)$ is the mth iterate defined by (14) and the subscript indicates dependence on $u(\cdot)$. We now present an application of the operator $\mathbb{N}$ to dynamic inversion.

III. Regulation Based on Nonlinear Inversion

In the following section we describe an application of the nonlinear operator $\mathbb{N}$ to nonlinear dynamic inversion. The resulting input is used to develop a nonlinear regulator. Our approach is based on the generation of the triplet $(x_d(\cdot), u_d(\cdot), y_d(\cdot))$, through nonlinear inversion, where $y_d(\cdot)$ is the desired output trajectory, and $x_d(\cdot)$ and $u_d(\cdot)$ are the corresponding desired input and state trajectories which yield the desired output trajectory. The critical issue is to find a bounded (possibly noncausal) $u_d(\cdot)$. We begin this section by formulating this inversion problem.

[1] We make formal use of the dirac-δ function in (27). This can be eliminated with a longer argument using our definition of $\phi(t)$ (3).

A. Stable Inversion Problem

Consider the nonlinear system

$$\dot{x}(t) = f[x(t)] + g[x(t)]u(t) \tag{28}$$
$$y(t) = h[x(t)] \tag{29}$$

defined on a neighborhood X of the origin of $\mathbb{R}^n$ with input $u(\cdot) \in \mathbb{R}^q$ and output $y(\cdot) \in \mathbb{R}^q$. The functions $f(x)$, $g_i(x)$ (the ith column of $g(x)$) $i = 1, 2, \cdots, q$ are smooth vector fields, and $h_i(x)$ for $i = 1, 2, \cdots, q$ are smooth functions on X with $f(0) = 0$ and $h(0) = 0$.

In the context of the above system, pose the following.

Stable Inversion Problem: Given a smooth reference output trajectory $y_d(\cdot) \in L_1 \cap L_\infty$, find a control input $u_d(\cdot)$ and a state trajectory $x_d(\cdot)$ such that:

1) $u_d(\cdot)$ and $x_d(\cdot)$ satisfy the differential equation

$$\dot{x}_d(t) = f[x_d(t)] + g[x_d(t)]u_d(t). \tag{30}$$

2) Exact output tracking is achieved

$$h[x_d(t)] = y_d(t). \tag{31}$$

3) $u_d(\cdot)$ and $x_d(\cdot)$ are bounded, and

$$u_d(t) \to 0,\ x_d(t) \to 0 \qquad \text{as } t \to \pm\infty. \tag{32}$$

We call $x_d(\cdot)$ the desired state trajectory, and $u_d(\cdot)$ the nominal control input. These can be incorporated into a regulator by using u_d as a feedforward signal, and $x(\cdot) - x_d(\cdot)$ as an error signal for feedback (see Fig. 2).

B. Application of $\mathbb{N}$ to Stable Inversion

1) Mathematical Preliminaries: In solving for the trajectories $x_d(\cdot)$ and $u_d(\cdot)$, the concepts of stable and unstable manifolds of an equilibrium point arise naturally [19]. For the sake of completeness we review the definitions here. Let $z = 0$ be an equilibrium point of an autonomous system defined in an open neighborhood $\overline{U}$ of the origin of $\mathbb{R}^n$

$$\dot{z}(t) = f[z(t)] \tag{33}$$

and $\overline{\phi}_t(z)$ be the flow passing through z at $t = 0$. We define the (local) stable and unstable manifolds W^s, W^u as follows:

$$W^s = \{z \in \overline{U} | \overline{\phi}_t(z) \in \overline{U}\, \forall t \geq 0,\ \overline{\phi}_t(z) \to 0 \text{ as } t \to \infty\} \tag{34}$$

$$W^u = \{z \in \overline{U} | \overline{\phi}_t(z) \in \overline{U}\, \forall t \leq 0,\ \overline{\phi}_t(z) \to 0 \text{ as } t \to -\infty\}. \tag{35}$$

The equilibrium point $z = 0$ is said to be hyperbolic if the Jacobian matrix $D_z f$ at $z = 0$ has no eigenvalues on the $j\omega$ axis. Let n^s denote the number of eigenvalues of $D_z f$ in the open left-half complex plane and n^u the number in the open right half-plane. Stable and unstable manifolds W^s and W^u exist locally in the neighborhood of hyperbolic fixed points and have dimensions n^s and n^u, respectively.

For convenience, we will use the following notation. Let $\underline{N} \triangleq \{0, 1, 2, \cdots\}$, $r = (r_1, r_2, \cdots, r_q)^T \in \underline{N}^q$ and $y(t) = (y_1(t), y_2(t), \cdots, y_q(t))^T$; $t \in \mathbb{R}$. Then we define $|r| \triangleq r_1 + r_2 + \cdots + r_q$ and write

$$y^{(r)}(t) \triangleq \begin{bmatrix} \frac{d^{r_1} y_1}{dt^{r_1}}(t) \\ \frac{d^{r_2} y_2}{dt^{r_2}}(t) \\ \cdots \\ \frac{d^{r_q} y_q}{dt^{r_q}}(t) \end{bmatrix}. \tag{36}$$

If y: $\mathbb{R}^n \to \mathbb{R}^q$ and f: $\mathbb{R}^n \to \mathbb{R}^n$, we define

$$L_f^r(y)(t) \triangleq \begin{bmatrix} L_f^{r_1} y_1(t) \\ L_f^{r_2} y_2(t) \\ \cdots \\ L_f^{r_q} y_q(t) \end{bmatrix}. \tag{37}$$

2) Partial Linearization and Inversion: The system dynamics, (28) and (29), are written in the following form, where the number of inputs (q) is assumed to be the same as the number of outputs

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{q} g_i(x(t))u_i(t) \tag{38a}$$

$$y_1(t) = h_1(x(t))$$
$$\cdots$$
$$y_q(t) = h_q(x(t)). \tag{38b}$$

We assume that the system has well-defined relative degree $r = (r_1, r_2, \cdots, r_q)$ at the equilibrium point zero, that is, 1) for all $1 \leq j \leq q$, for all $1 \leq i \leq q$, for all $k < r_i - 1$, and for all x in a neighborhood of the origin

$$L_{g_j} L_f^k h_i(x) = 0 \tag{39}$$

and 2) the $q \times q$ matrix

$$\beta(x) = \begin{bmatrix} L_{g_1} L_f^{r_1 - 1} h_1(x) & \cdots & L_{g_q} L_f^{r_1 - 1} h_1(x) \\ L_{g_1} L_f^{r_2 - 1} h_2(x) & \cdots & L_{g_q} L_f^{r_2 - 1} h_2(x) \\ \cdots & \cdots & \cdots \\ L_{g_1} L_f^{r_q - 1} h_q(x) & \cdots & L_{g_q} L_f^{r_q - 1} h_q(x) \end{bmatrix} \tag{40}$$

is nonsingular in a neighborhood of the origin.

Under this assumption, the system can be partially linearized. To do this, we differentiate $y_i(\cdot)$ until at least one $u_j(t)$ appears explicitly. This will happen at exactly the r_ith derivative of $y_i(\cdot)$ due to (39). Define $\xi_k^j(t) = y_i^{(k-1)}(t)$ for $i = 1, \cdots, q$ and $k = 1, \cdots, r_i$, and denote

$$\begin{aligned} \xi(t) &= (\xi_1^1(t), \xi_2^1(t), \cdots, \xi_{r_1}^1(t), \xi_1^2(t), \cdots, \xi_{r_2}^2(t), \cdots, \xi_{r_q}^q(t))^T \\ &= (y_1(t), \dot{y}_1(t), \cdots, y_1^{(r_1 - 1)}(t), y_2(t), \cdots, y_2^{(r_2 - 1)}(t), \cdots, \\ &\quad y_q^{(r_q - 1)}(t))^T. \end{aligned} \tag{41}$$

Choose η, an $n - |r|$-dimensional function on $\mathbb{R}^n$ such that $(\xi^T, \eta^T)^T = \psi(x)$ forms a change of coordinates with $\psi(0) = 0$ [19]. In these new coordinates, the system dynamics (28) becomes

$$\begin{cases} \dot{\xi}_1^i(t) = \xi_2^i(t) \\ \cdots \\ \dot{\xi}_{r_i - 1}^i(t) = \xi_{r_i}^i(t) \\ \dot{\xi}_{r_i}^i(t) = \alpha_i[\xi(t), \eta(t)] + \beta_i[\xi(t), \eta(t)]u(t) \end{cases} \quad \text{for } i = 1, \cdots, q \tag{42a}$$

$$\dot{\eta}(t) = s_1[\xi(t), \eta(t)] + s_2[\xi(t), \eta(t)]u(t) \tag{42b}$$

which, in a more compact form, is equivalent to

$$y^{(r)}(t) = \alpha[\xi(t), \eta(t)] + \beta[\xi(t), \eta(t)]u(t) \tag{43a}$$

$$\dot{\eta}(t) = s_1[\xi(t), \eta(t)] + s_2[\xi(t), \eta(t)]u(t) \tag{43b}$$

where

$$y(t) = (y_1(t), y_2(t), \cdots, y_q(t))^T$$

$$u(t) = (u_1(t), u_2(t), \cdots, u_q(t))^T$$

$$\alpha[\xi(t), \eta(t)] = L_f^r h[\psi^{-1}(\xi(t), \eta(t))] \tag{44}$$

$$\beta[\xi(t), \eta(t)] = L_g^1 L_f^{r-1} h[\psi^{-1}(\xi(t), \eta(t))]. \tag{45}$$

Here β is as defined in (40), $\alpha(0, 0) = 0$ since $f(0) = 0$, and

$$h(x(t)) = [h_1(x(t)), h_2(x(t)), \cdots, h_q(x(t))]^T$$

$$g(x(t)) = [g_1(x(t)), g_2(x(t)), \cdots, g_q(x(t))].$$

Since by the relative degree assumption, $\beta(\xi(t), \eta(t))$ is nonsingular, the following feedback control law:

$$u(t) \triangleq \beta^{-1}(\xi(t), \eta(t))[v(t) - \alpha(\xi(t), \eta(t))] \tag{46}$$

is well defined and partially linearizes the system such that the input–output relationship is given by q chains of integrators

$$y^{(r)}(t) = v(t) \tag{47}$$

where $v(t) \in \mathbb{R}^q$ is the new control input. To maintain exact tracking choose

$$v(t) = y_d^{(r)}(t). \tag{48}$$

Then

$$\xi(t) = \xi_d(t) \triangleq [y_{d1}(t), \dot{y}_{d1}(t), \cdots, y_{d1}^{(r_1-1)}(t), y_{d2}(t), \cdots, y_{d2}^{(r_2-1)}(t), \cdots, y_{dq}^{(r_q-1)}(t)]^T \tag{49}$$

and (43b) becomes (50), which we call the *internal dynamics*, or the zero dynamics driven by the reference output trajectory

$$\dot{\eta}(t) = s[\xi_d(t), \eta(t), y_d^{(r)}(t)] \triangleq s(\eta(t), Y_d(t)) \tag{50}$$

where

$$\begin{aligned} s[\eta(t), Y_d^{(r)}(t)] &\triangleq s_1(\xi_d(t), \eta(t)) + s_2(\xi_d(t), \eta(t))\beta^{-1}(\xi_d(t), \eta(t)) \\ &\quad \times [y_d^{(r)}(t) - \alpha(\xi_d(t), \eta(t))] \end{aligned} \tag{51}$$

where $Y_d(\cdot)$ represents $\xi_d(\cdot)$ and $y_d^{(r)}(\cdot)$.

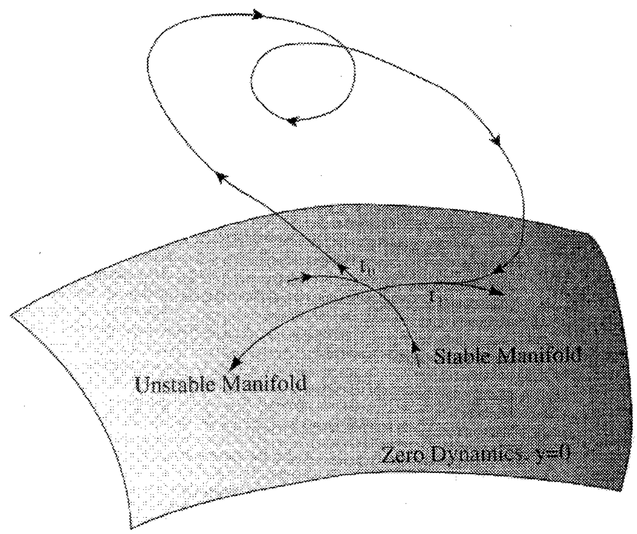

Fig. 1. Geometric interpretation of stable inversion.

3) Application of Theorem 1 to Inversion: If s satisfies Condition 1, then by Theorem 1 there exists a solution $\eta_d(\cdot) \in L_1 \cap L_\infty \cap C^0$ to (50). Once the solution to the internal dynamics is found, the original state trajectory is given by the inverse coordinate transformation

$$x_d(t) = \Psi^{-1}\begin{pmatrix}\xi_d(t)\\ \eta_d(t)\end{pmatrix} \tag{52}$$

and an input trajectory, $u_d(t)$, by (46)

$$\begin{aligned} u_d(t) &= \beta^{-1}(\xi_d(t), \eta_d(t))[-\alpha(\xi_d(t), \eta_d(t))] \\ &\triangleq \hat{\beta}^{-1}(Y_d(t), \eta_d(t))[-\hat{\alpha}(Y_d(t), \eta_d(t))]. \end{aligned} \tag{53}$$

4) Geometric Interpretation of Stable Inversion: If $Y_d(\cdot)$ has a compact support, $[t_0, t_1]$, then it is possible to give a geometric interpretation of the evolution of $x_d(t)$ [17]. The noncausal part of the nominal control drives the internal states of the system along the unstable manifold of the zero dynamics manifold to a particular initial condition $x_d(t_0)$ while maintaining zero system output. This initial condition guarantees two things: 1) the desired reference output trajectory is easily reproduced with bounded input and states, and 2) the system state "lands on" the stable manifold of the zero dynamics manifold at the end of output tracking. With this nice final condition, the internal states will converge to zero along the stable manifold without affecting the output. This geometrical picture is shown in Fig. 1.

5) Contrast with Hirschorn's Inverse: In the partially linearized normal form

$$y^{(r)}(t) = \alpha(\xi(t), \eta(t)) + \beta(\xi(t), \eta(t))u(t) \tag{43a}$$

$$\dot{\eta}(t) = s_1(\xi(t), \eta(t)) + s_2(\xi(t), \eta(t))u(t) \tag{43b}$$

the comparison between the stable inversion technique and that of Hirschorn can be made clear. Solve the first equation for $u(\cdot)$, that is, $u(t) \triangleq \beta^{-1}(\xi(t), \eta(t))[v(t) - \alpha(\xi(t), \eta(t))]$

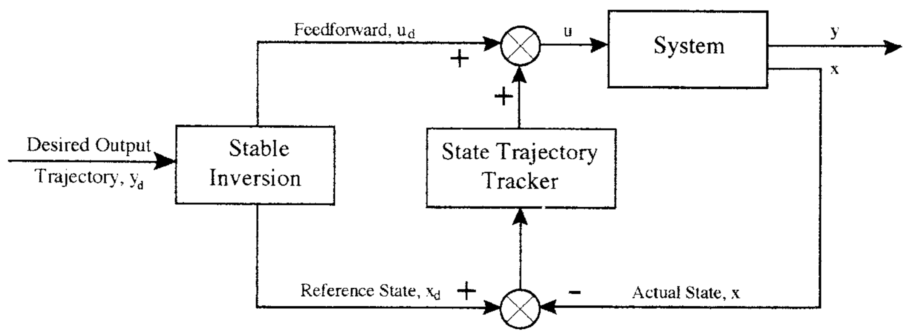

Fig. 2. Block diagram of the control scheme.

[see (47)] and viewing $(y^{(r)}(\cdot), \xi(\cdot))$ as the input, this "output equation" for $u(\cdot)$ along with

$$\dot{\eta}(t) = s(\xi_d(t), \eta(t), y_d^{(r)}(t)); \qquad \eta(0) = 0$$

forms Hirschorn's left-inverse system. This inverse is realizable with standard integrators running forward in time but could lead to unbounded solutions for nonminimum phase systems. In contrast, when the initial condition $\eta(0) = 0$ is replaced by the boundary conditions $\eta(\pm\infty) = 0$, to yield our inverse, the system in no longer realizable with integrators running forward in time. Rather, causal and noncausal linear filters are used iteratively to find η and u. Note that when the linearization is causal, the iterations also result in a causal operator output which coincides with Hirschorn's solution.

C. Regulator Based on Nonlinear Inversion

Let $(u_d(\cdot), x_d(\cdot))$ be a bounded preimage of $y_d(\cdot)$, found through the inverse operator, to system (28) and (29). We establish below that the system's state and output exponentially track the state trajectory $x_d(\cdot)$ and the corresponding desired output $y_d(\cdot)$.

Lemma 5: Given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that $\|u_d(\cdot)\|_\infty < \epsilon$ if $[\|Y_d(\cdot)\|_1 + \|Y_d(\cdot)\|_\infty] < \delta$.

Proof: To find the inverse input-state trajectory, a bounded solution to the internal dynamics [see (50)] is found iteratively as the fixed point of the operator P_{Y_d}. It follows from (21) that this fixed point is bounded

$$\|\eta_d(\cdot)\|_\infty \leq \frac{K_2\|\phi(\cdot)\|_\beta\|Y_d(\cdot)\|_1}{1 - K_1\|\phi(\cdot)\|_\alpha} \leq r. \tag{54}$$

From (53)

$$u_d(t) = \hat{\beta}^{-1}(Y_d(t), \eta_d(t))[-\hat{\alpha}(Y_d(t), \eta_d(t))].$$

The lemma follows since $\hat{\alpha}(0, 0) = 0$, $\hat{\beta}^{-1}$ is bounded, and $\hat{\alpha}$ is continuous [19]. □

Proposition 1: Let $(u_d(\cdot), x_d(\cdot))$ be a bounded input-state trajectory in the preimage of $y_d(\cdot)$ for (28) and (29) $D_x f|_0$ be Hurwitz, and $\|Y_d(\cdot)\|_1 + \|Y_d(\cdot)\|_\infty$ small. Then, the desired state trajectory $x_d(\cdot)$ is bounded, and the output $y(t)$ tends to the desired output $y_d(t)$ asymptotically.

Proof: This follows from standard Lyapunov arguments based on the linearization at $x = 0$ [18]. □

If the origin of the system to be controlled is unstable, but is stabilizable in the first approximation, then we use a control law of the form

$$u(t) = u_d(t) + K(x_d(t) - x(t))$$

where $K(x_d(t) - x(t)$ is the stabilizing feedback, with $D_x f|_0 - Kx$ Hurwitz, and $u_d(\cdot)$ is the feedforward (see Fig. 2). Here, we used standard stabilizability in first approximation (it is not our objective to present any new stabilization method). Other techniques for stabilization can be found in [19] and [20]. For a global approach see [21]. Note that the stable inversion technique only provides a nominal feedforward input $u_d(\cdot)$ and a desired state-trajectory $x_d(\cdot)$ which is to be stabilized. The input-state trajectory is predetermined by the system, for a given desired-output trajectory, and does not depend on the initial conditions or the stabilizing feedback. This concludes the formulation of the nonlinear inversion-based regulator.

IV. An Example

In this section, both the inversion-based regulator and Byrnes–Isidori regulator approaches will be applied to a simple nonlinear nonminimum-phase system. The example system is selected such that the solution to the nonlinear partial differential equations resulting from the Byrnes–Isidori regulator approach is manageable. The performances of the two approaches are compared.

Consider the single-input single-output system described by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -x_1(t) + x_2(t) \\ -3x_2(t) + x_1^3(t) \\ x_1(t) - 2x_3(t) \\ -x_4(t) + x_3^2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 + \sin^2(x_4(t)) \\ 0 \\ 0 \end{bmatrix} u(t) \tag{55}$$

$$y(t) = x_1(t) - 3x_3(t) \tag{56}$$

and a reference output trajectory given by

$$y_d(t) = \begin{cases} \frac{A_m}{2}(1 - \cos(t)) & t \in [0, 2\pi] \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

where $\frac{A_m}{2}$ is 0.1. Note that $\frac{A_m}{2}$ has to be small enough to keep the internal dynamics local to the origin.

A. The Byrnes–Isidori Regulator

First, let us consider the Byrnes–Isidori regulator approach. The reference signal can be exactly generated by the following linear time-invariant exosystem:

$$\begin{aligned}\dot{w}_1(t) &= w_2(t)\\ \dot{w}_2(t) &= -w_1(t)\\ \dot{w}_3(t) &= 0\\ y_d &= w_1(t) + w_3(t)\end{aligned} \tag{58}$$

with the initial conditions set at $t = 0$, and reset at $t = 2^*\pi$, as follows:

$$\begin{gathered} w_1(-\infty) = w_2(-\infty) = w_3(-\infty) = 0\\ w_1(0) = -\frac{A_m}{2}, \qquad w_3(0) = \frac{A_m}{2}, \qquad w_2(0) = 0\\ w_1(2\pi) = w_2(2\pi) = w_3(2\pi) = 0. \end{gathered} \tag{59}$$

The zero error manifold, $x = x(w)$ and $u = u(w)$, is obtained by solving a system of nonlinear partial differential equations which is, in general, extremely difficult if not impossible. For this example, the partial differential equations are as follows:

$$\begin{aligned} &\frac{\partial x_1(w)}{\partial w_1} w_2 - \frac{\partial x_1(w)}{\partial w_2} w_1 = -x_1(w) + x_2(w)\\ &\frac{\partial x_2}{\partial w_1} w_2 - \frac{\partial x_2}{\partial w_2} w_1\\ &\qquad = -3x_2(w) + x_1^3(w) + (2 + \sin^2 x_4(w))u(w)\\ &\frac{\partial x_1(w)}{\partial w_1} w_2 - \frac{\partial x_1(w)}{\partial w_2} w_1 = x_1(w) - 2x_3(w)\\ &\frac{\partial x_1(w)}{\partial w_1} w_2 - \frac{\partial x_1(w)}{\partial w_2} w_1 = -x_4(w) + x_3^2(w) \end{aligned} \tag{60}$$

subject to

$$x_1(w) - 3x_3(w) = w_1 + w_3. \tag{61}$$

We chose the system such that (60) and (61) have a closed-form solution. This is given by

$$\begin{aligned} x_1(w) &= -\frac{1}{2}w_1 - \frac{3}{2}w_2 - 2w_3\\ x_2(w) &= w_1 - 2w_2 - 2w_3\\ x_3(w) &= -\frac{1}{2}(w_1 + w_2) - w_3\\ x_4(w) &= \frac{7}{20}w_1^2 + \frac{3}{20}w_2^2 + \frac{1}{10}w_1 w_2 + w_1 w_3 + w_3^2\\ u(w) &= (2w_1 + w_2 + 3x_2(w) - x_1^3(w))/(2 + \sin^2 x_4(w)). \end{aligned} \tag{62}$$

Note that with $u(t) = 0$ (undriven system), the origin is locally asymptotically stable since the Jacobian matrix of $f(x)$ at $x = 0$ is clearly

$$\begin{bmatrix} -1 & 1 & 0 & 0\\ 0 & -3 & 0 & 0\\ 1 & 0 & -2 & 0\\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{63}$$

and has all its eigenvalues negative. Therefore, for simplicity, we can choose the feedback gain to be zero.

This completes the Byrnes–Isidori regulator design. The simulation results are presented for a given output trajectory shown in Fig. 3. The desired state trajectories [obtained from (62)] which yield exact output tracking are shown in Fig. 4. Fig. 5 shows the actual state trajectories which differ from the desired ones. The resultant output trajectory is shown in Fig. 3 (solid line). Note that the output generated by the regulator does asymptotically track the reference trajectory as predicted by theory. This is evidenced by the segments from $t = 3$ to $t = 2\pi$ and $t > 8$. However, there are substantial transient tracking errors (see Fig. 3) both when getting onto the zero error manifold and when changing the manifold.

B. Nonlinear Inversion-Based Regulator

Next, we consider the stable inversion approach. To partially linearize the system, we differentiate the output y to yield

$$\dot{y}(t) = \dot{x}_1(t) - 3\dot{x}_3(t) = -4x_1(t) + x_2(t) + 6x_3(t). \tag{64}$$

Since the control $u(\cdot)$ does not appear explicitly, we differentiate $\dot{y}(\cdot)$ again to yield

$$\begin{aligned} \ddot{y}(t) &= -4(-x_1(t) + x_2(t)) - 3x_2(t) + x_1^3(t)\\ &\quad + (2 + \sin^2(x_4(t)))u(t) + 6(x_1(t) - 2x_3(t))\\ &= (10x_1(t) - 7x_2(t) - 12x_3(t) + x_1^3(t))\\ &\quad + (2 + \sin^2(x_4(t)))u(t) \triangleq \alpha(x(t)) + \beta(x(t))u(t). \end{aligned} \tag{65}$$

Now, not only does $u(\cdot)$ appear, its coefficient $\beta(x)$ is nonzero for all x. Hence, we can set

$$u(x(t)) = \frac{1}{\beta(x(t))}[\ddot{y}_d(t) - \alpha(x(t))] \tag{66}$$

and introduce a change of coordinates

$$\begin{bmatrix} y\\ \dot{y}\\ \eta_1\\ \eta_2 \end{bmatrix} = \begin{bmatrix} y\\ \dot{y}\\ x_3\\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 0\\ -4 & 1 & 6 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix} \triangleq \psi(x). \tag{67}$$

The inverse transformation is given by

$$\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 0\\ 4 & 1 & 6 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y\\ \dot{y}\\ \eta_1\\ \eta_2 \end{bmatrix}. \tag{68}$$

Using the feedback of (66), the system in the new coordinates becomes

$$\begin{aligned} \ddot{y}(t) &= \ddot{y}_d(t)\\ \dot{\eta}(t) &= s(\eta(t), y_d(t)) \end{aligned} \tag{69}$$

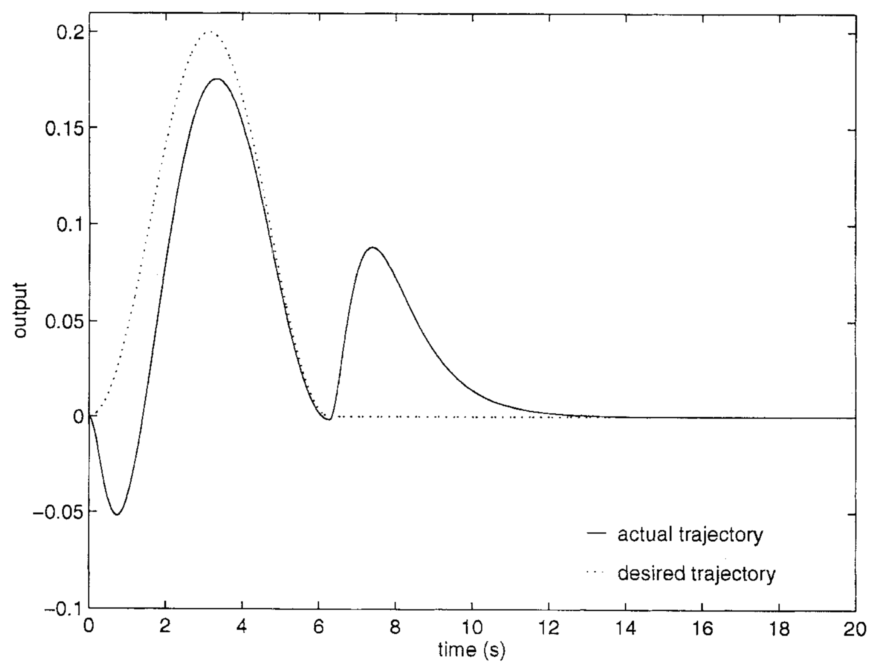

Fig. 3. Output trajectory for the Byrnes–Isidori regulator.

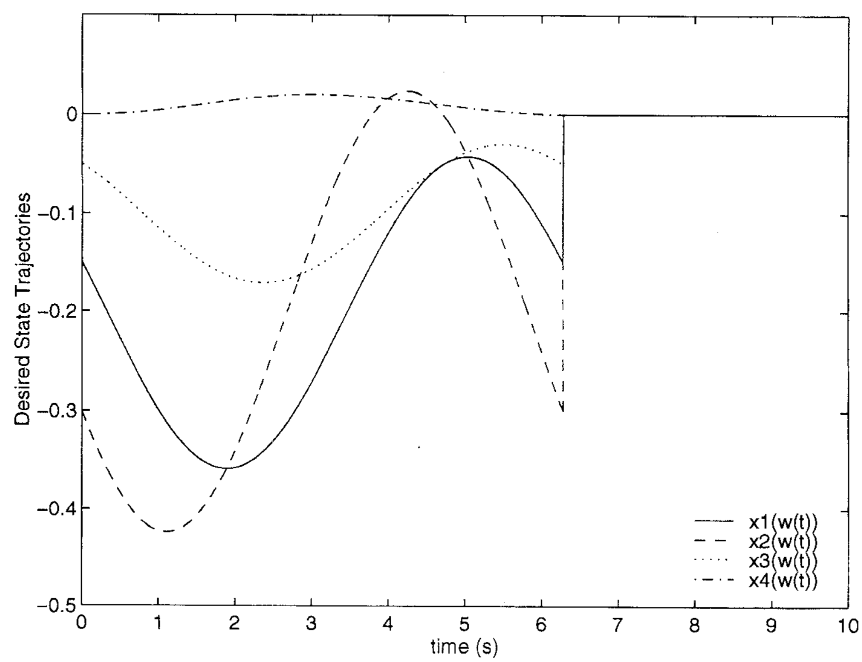

Fig. 4. Desired state trajectory for the Byrnes–Isidori regulator.

where

$$\eta \triangleq (\eta_1 \quad \eta_2)^T$$

and

$$s(\eta, y_d) \triangleq \begin{bmatrix} \eta_1 + y_d \\ -\eta_2 + \eta_1^2 \end{bmatrix}. \tag{70}$$

We now show that $s(\cdot, \cdot)$ satisfies Condition 1. Hence, by Theorem 1, there exists a solution $\eta_d(\cdot)$ to the above equation which belongs to $L_1 \cap L_\infty \cap C^0$.

Let A be the Jacobian of s

$$A \triangleq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{71}$$

Equation (3) yields the following state transition matrix:

$$\phi_A(t) \triangleq \begin{bmatrix} 0 & 0 \\ 0 & e^{-t} \end{bmatrix} \quad \text{if} \quad t > 0$$

$$\phi_A(t) \triangleq \begin{bmatrix} -e^{t} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{if} \quad t < 0. \tag{72}$$

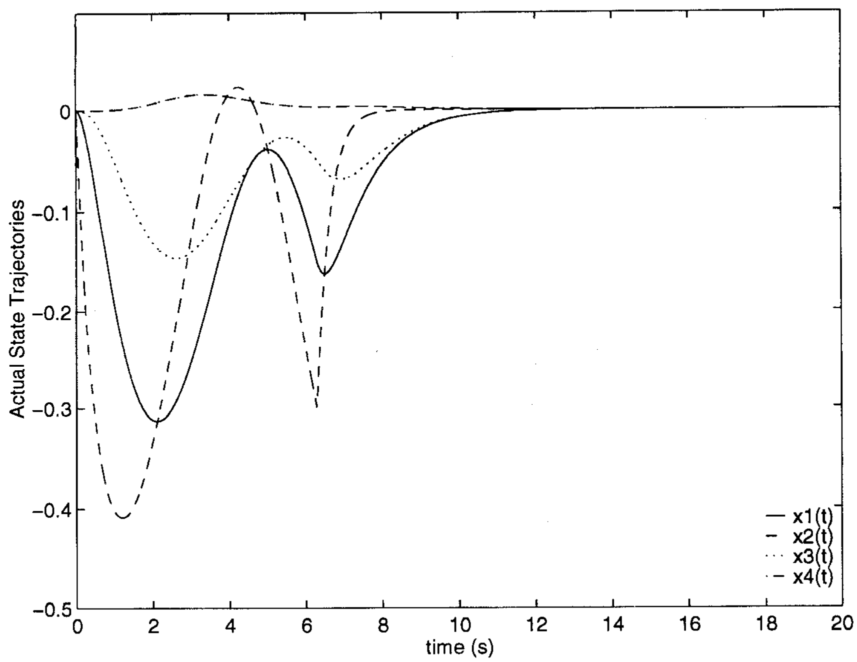

Fig. 5. Actual state trajectory for the Byrnes–Isidori regulator.

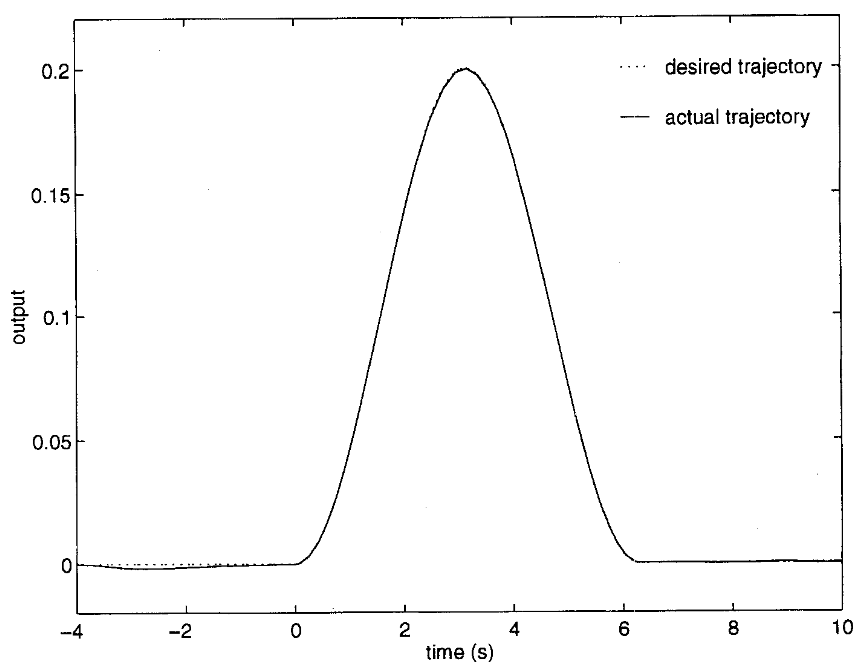

Fig. 6. Output trajectory for the inversion-based regulator.

From (6) and (7) we also obtain that $\|\phi_A(\cdot)\|_\alpha = 2$ and $\|\phi_A(\cdot)\|_\beta = 2$. Choose K_1 such that $0 < K_1 < 0.5$, $r \leq K_1/2$, and $K_2 = 1$. With these values for A, K_1, K_2, and r, it is easy to verify that s satisfies (10)–(12) and thereby Condition 1. Hence, there exists a solution to (70) given by $\eta_d(\cdot) = \mathbb{N}(y_d(\cdot))$.[2]

Once $\eta_d(\cdot)$ is calculated, the desired trajectory in the original coordinates can be calculated using the inverse coordinate transformation so that

$$x_d(t) = \psi^{-1}[y_d(t), \dot{y}_d(t), \eta_{1d}(t), \eta_{2d}(t)]$$

$$u(x_d(t)) = \frac{1}{\beta(x_d(t))}[\ddot{y}_d(t) - \alpha(x_d(t))]$$

where $\psi^{-1}(\cdot)$ is given by (68) and the nominal feedforward input $u_d(t)$ is calculated according to the linearized feedback law (66). Note that since $\eta_d(t)$ is nonzero for $t < t_0$, the

[2] Matlab routines can be obtained by email to santosh@stress.mech.utah.edu.

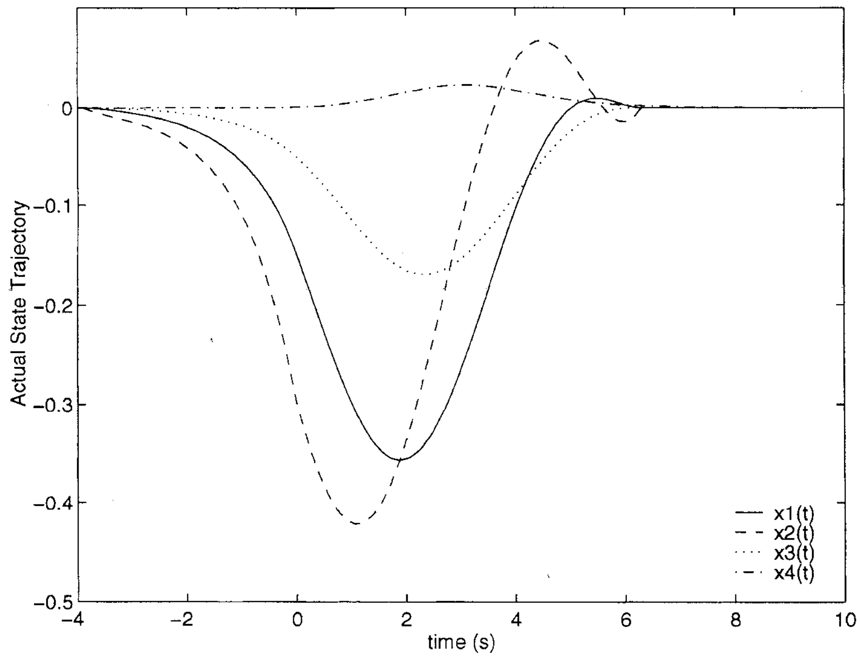

Fig. 7. Actual state trajectory for the inversion-based regulator.

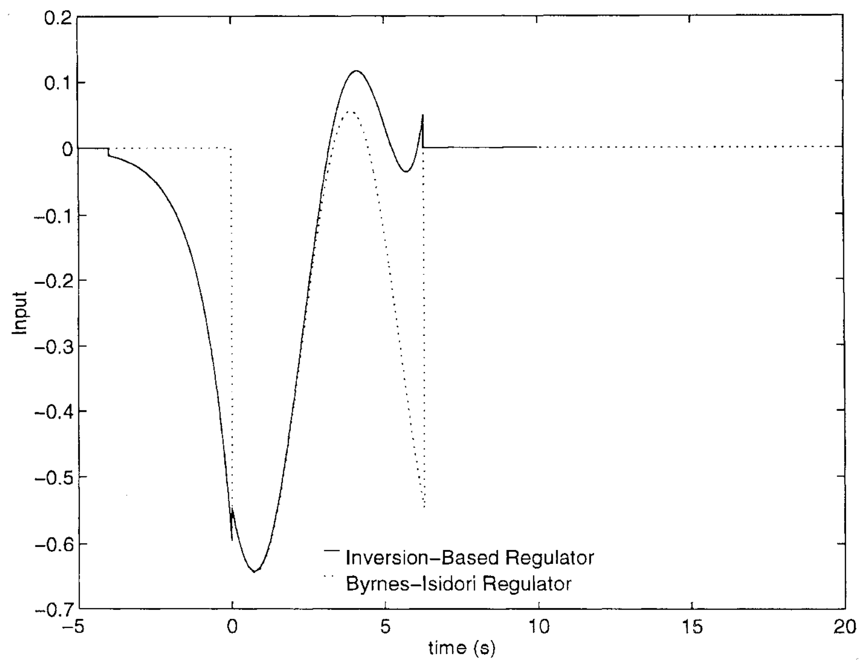

Fig. 8. Comparison of inputs to the regulators.

corresponding $u_d(t)$ is also nonzero for $t < t_0$. This leads to a noncausal feedforward input.

As in the case of the Byrnes–Isidori regulator, we choose zero feedback gain because the linear approximation of the undriven system is asymptotically stable. Simulation results are shown in Fig. 6 which shows that the actual trajectory closely matches the desired output trajectory (not absolutely exact because the input is truncated at $t = -4$). Fig. 7 shows the corresponding actual state trajectories, generated by the truncated input (Fig. 8). Fig. 8 also compares the input used in the inversion-based regulator to the input for the Byrnes–Isidori case. Note that the inputs are of the same order of magnitude. Hence, it is advantageous to use the proposed inversion-based regulator because an almost perfect output tracking (compare Figs. 3 and 6) is obtained with a similar control effort. If the input generated by the inversion-based regulator is truncated, then the resulting transient errors are similar as in the Byrnes–Isidori case (see Fig. 9); however,

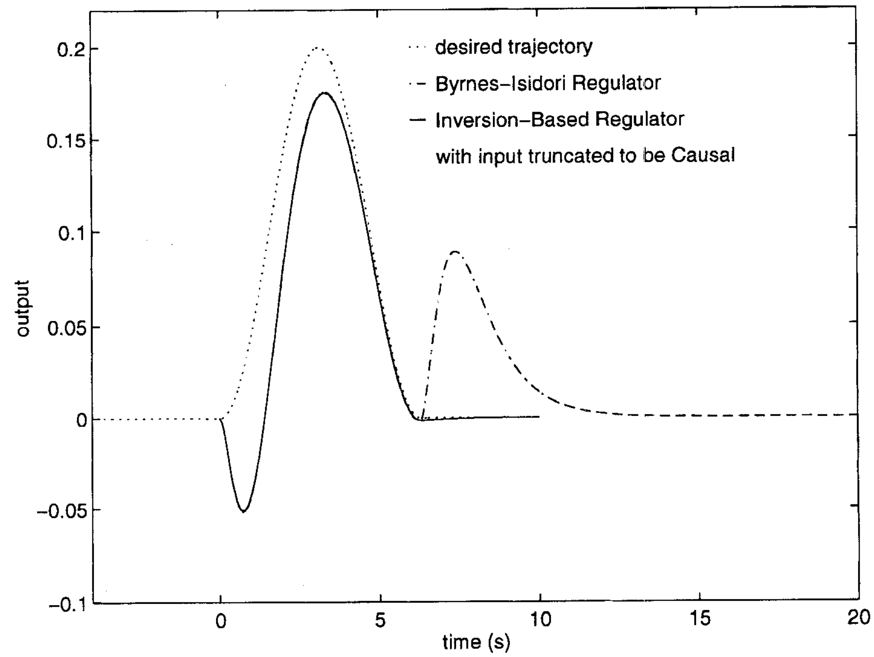

Fig. 9. Comparison of output trajectory.

the errors at the end of the motion $t = 2\pi$ present in the Byrnes–Isidori regulator (due to switching in the exosystem) are absent. This is mainly due to the anticipatory or noncausal nature of the inverse-based regulator—when truncated to be causal transients are present at $t = 0$, but quite reduced when anticipated output changes occur at $t = 2\pi$. A small tracking error is still noticeable which is absent when no truncations are present; this reflects the rate of convergence to the desired state-trajectory and the size of the initial error.

V. Conclusions

We have introduced a nonlinear operator whose application in nonlinear inversion yields a clear connection between unstable zero dynamics and noncausal inversion. When noncausal inversion is incorporated into tracking regulators, we can see that it is a powerful tool for control—particularly when computation is considered. An important fact is that a given system model defines different input–output operators depending on how boundary conditions are applied. For the study of feedforward control, boundary conditions at infinity give a useful perspective on a system. We have considered only the case of hyperbolic zero dynamics. Cases where zero dynamics have a center manifold or a hyperbolic orbit should prove interesting as well.

Appendix

If $x(\cdot)$ is a vector-valued measurable function where $x(t) = [x_1(t), x_2(t), x_3(t), \cdots, x_n(t)]^T \in R^n$, then

$$\|x(t)\|_1 \triangleq \sum_{i=1}^{n} |x_i(t)|, \qquad \text{the standard "1" norm in } R^n$$

$$\|x\|_1 = \|x(\cdot)\|_1 \triangleq \int_{-\infty}^{\infty} \|x(t)\|_1 \, dt$$

$$\|x(t)\|_\infty \triangleq \max_i |x_i(t)| \text{ is the standard } \infty \text{ norm in } R^n$$

$$\|x\|_\infty = \|x(\cdot)\|_\infty \triangleq \operatorname{ess\,sup}_{t\in(-\infty,\,\infty)} \|x(t)\|_\infty$$

$$\|x(t)\|_U \triangleq \|x(t)\|_1 + \|x(t)\|_\infty$$

$$\|x\|_U \triangleq \|x\|_1 + \|x\|_\infty.$$

If h_1: $R^n \to R^m$, h_2: $R^n \to R^n$, and $x \in R^n$, then

$$L_{h_2} h_1 = \frac{dh_1}{dx} h_2(x)$$

$$L_{h_2}^n h_1 = L_{h_2}(L_{h_2}^{n-1} h_1).$$

$h_1(x) = O(\|x\|_n) \Rightarrow \frac{\|h_1(x)\|_m}{\|x\|_n}$ is bounded in a neighborhood of $x = 0$, where $\|\cdot\|_m$ and $\|\cdot\|_n$ are norms in R^m and R^n, respectively.

References

[1] B. D. O. Anderson and J. Moore, *Optimal Control-Linear Quadratic Methods.* Englewood Cliffs, NJ: Prentice-Hall, 1990.
[2] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, vol. 12, pp. 457–465, 1976.
[3] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 131–140, 1990.
[4] J. Huang and W. J. Rugh, "On a nonlinear multivariable servomechanism problem," *Automatica*, vol. 26, no. 6, pp. 963–972, 1990.
[5] A. J. Krener, "Optimal model matching controllers for linear and nonlinear systems," in *Proc. NOLCOS*, Bordeaux, France, 1992.
[6] S. Gopalswamy and J. K. Hedrick, "Tracking nonlinear nonminimum phase systems using sliding control," *Int. J. Contr.*, vol. 57, no. 5, pp. 1141–1158, 1993.
[7] B. Castillo, "Output regulation of nonlinear systems with more inputs than outputs," *Int. J. Contr.*, vol. 57, no. 6, pp. 1343–1356, 1993.
[8] R. M. Hirschorn, "Invertibility of nonlinear control systems," *SIAM J. Contr. Optimiz.*, vol. 17, no. 2, pp. 289–297, 1979.

[9] M. D. Di Benedetto and P. Lucibello, "Inversion of nonlinear time-varying systems," *IEEE Trans. Automat. Contr.*, vol. 38, no. 8, pp. 1259–1264, Aug. 1993.
[10] J. Hauser, S. Sastri, and G. Meyer, "Nonlinear control design for slightly nonminimum phase systems: Application to V/STOL aircraft," *Automatica*, vol. 28, pp. 665–679, 1992.
[11] R. Gurumoorthy and S. R. Sanders, "Controlling nonminimum phase nonlinear systems—The inverted pendulum on a cart example," in *Proc. Amer. Contr. Conf.*, 1993, pp. 680–685.
[12] E. Bayo, "A finite-element approach to control the end-point motion of single-link flexible robot," *J. Robot. Syst.*, vol. 4, no. 1, pp. 63–75, 1987.
[13] B. Paden, D. Chen, R. Ledesma, and E. Bayo, "Exponentially stable tracking control for multi-joint flexible-link manipulators," *ASME J. Dynamic Syst., Meas. Contr.*, vol. 115, pp. 53–59, 1993.
[14] G. Meyer, L. R. Hunt, and R. Su, "Nonlinear system guidance," submitted to *CDC*, 1995.
[15] J. K. Hale, *Ordinary Differential Equations*, vol. 21. New York: Wiley-Interscience, 1980.
[16] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*. New York: McGraw-Hill, 1978.
[17] B. Paden and D. Chen, "A state-space condition for the invertibility of nonlinear nonminimum-phase systems," in *Proc. ASME Winter Annu. Meet.*, Anaheim, CA, 1992, DSC-vol. 43, pp. 37–41.
[18] H. K. Khalil, *Nonlinear Systems*. New York: Macmillan, 1992.
[19] A. Isidori, *Nonlinear Control Systems: An Introduction*. New York: Springer-Verlag, 1989.
[20] E. G. Albrecht, "On the optimal stabilization of nonlinear systems," *J. Appl. Math. Mech.*, vol. 25, pp. 1254–1266, 1961.
[21] E. D. Sontag, "Equilinearization: A simplified derivation and experimental results," in *Proc. Conf. Info. Sci. Syst.*, Johns Hopkins Univ., 1987, pp. 490–495.

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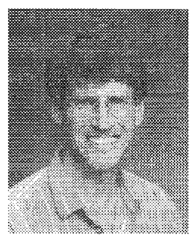

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