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Stable inversion of nonlinear non-minimum phase systems

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Output tracking control of non-minimum phase systems is a highly challenging problem encountered in the control of flexible manipulators, space structures, and elsewhere. Classical inversion provides exact output tracking but leads to internal instability, while recent nonlinear regulation provides stable asymptotic tracking but admits large transient errors. As a first step to solve this problem, this paper addresses the stable inversion of non-minimum phase nonlinear systems. Using the notions of zero dynamics and stable/unstable manifolds, an invertibility condition is established for a class of systems. A stable but non-causal inverse is obtained offline that can be incorporated into a stabilizing controller for dead-beat output tracking. This inverse contrasts with the causal inverse proposed by Hirschorn where unstable zero dynamics result in unbounded inverse solutions. Our results reduce to those of Hirschorn for minimum phase systems, however. In a numerical example, the stable inverse has achieved much superior tracking performance as compared with that produced using nonlinear regulation.

1. Introduction

Output tracking control of non-minimum phase systems is a highly challenging problem encountered in many practical engineering applications. An example that has received significant attention is the tip trajectory tracking problem for flexible-link manipulators (e.g. Paden *et al.* 1993, Zhao and Chen 1993, Cannon and Schmitz 1984). Other well-known non-minimum phase control problems include the aircraft altitude control problem, rocket trajectory tracking problem (Nichols *et al.* 1993), and many more. A system has non-minimum phase (or has unstable zeros in the linear case) if there exists a (nonlinear) state feedback that can hold the system output identically zero while the internal dynamics become unstable (Isidori 1989). The non-minimum phase property has long been recognized to be a major obstacle in many control problems. It is well-known that unstable zeros cannot be moved with state feedback while the poles can be arbitrarily placed (if completely controllable) (e.g. Wonham 1985). In standard adaptive control (e.g. Narendra and Annaswamy 1989) as well as in nonlinear adaptive control (e.g. Ghanadan and Blankenship 1993), all algorithms require that the plants have minimum phase. In the recent nonlinear control literature (e.g. Mahmoud and Khalil 1993), non-minimum phase is again a major barrier in feedback linearization and stabilization of nonlinear systems.

For output tracking of nonlinear systems, there are basically two approaches: classical inversion and output regulation, each having its own advantages and shortcomings. The classical inversion provides exact output tracking but leads to

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internal instability for non-minimum phase systems. This theory was first studied by Brockett and Mesarovic (1965) and more complete treatments were given by Sain and Massey (1969) and Silverman (1969). In these results, linear dynamical systems are considered mappings between input and output functions defined on $[0, \infty)$ with specific initial conditions. For a given output function, the input is obtained by solving the inverse system as an initial value problem. When the system has non-minimum phase, the initial value problem is unstable, leading to an unbounded inverse. The linear inversion results were extended by Hirschorn (1979) to real analytic nonlinear systems. Singh (1981, for example) had similar results on nonlinear inversion with modified conditions and considered their applications. Similar to the linear case, these inversion algorithms produce causal inverses for a given desired output $y_d(t)$ and a fixed initial condition $x(t_0)$, leading to unbounded $u(t)$ and $x(t)$ for non-minimum phase systems. This difficulty has been noted for a long time. Singh and Schy (1986) and De Luca and Siciliano (1989) have applied these inversion techniques to the control of flexible manipulators. Simulation and experimental results verify that, although exact output tracking can be achieved transiently, internal vibration builds up. Deeply rooted in these inversion results is the notion of relative degree that is also important in our work (although a clear exposition of relative degree (Isidori 1989) for nonlinear systems is fairly recent).

The nonlinear output regulation theory recently developed by Isidori and Byrnes (1990) ensures internal stability with asymptotic output tracking for a class of nonlinear systems with reference trajectories generated by an exosystem. The control law uses a feed-forward input plus feedback stabilization of a certain state trajectory. Both the feed-forward input and the state trajectory are obtained by solving a set of nonlinear partial differential equations (PDEs). This theory is a generalization of the corresponding linear results on the internal model principle by Francis and Wonham (1975) where a set of matrix equations can easily be solved to obtain the controller. The nonlinear regulator, however, encounters the difficulty of solving a set of nonlinear PDEs. Another problem with the nonlinear regulator is that transient errors cannot be controlled precisely and are usually large for non-minimum phase systems. This is verified by De Luca *et al.* (1990) through its application to flexible manipulator control. This general phenomenon is a fundamental limitation of the regulation approach.

Motivated by recent results on output tracking control of flexible-link manipulators by Paden *et al.* (1993) and Bayo *et al.* (1988), we introduce the notion of stable inversion and use it to develop a new approach for output tracking control of non-minimum phase systems. This new approach will provide stable asymptotic output tracking without transient errors, thus achieving the salient advantages of both classical inversion and output regulation. As in the regulation approach, there are two steps involved: (1) calculate a feed-forward control input and a desired state trajectory that maps into the desired output trajectory; and (2) stabilize the desired state trajectory using feedback control. The first step is the inversion step and is done offline (during the trajectory planning stage). The key to success is to guarantee stability of the feed-forward and state trajectories even for non-minimum phase systems. This is precisely the goal of this paper while the second step will not be studied here.

Beyond the application to output tracking control, the stable inversion problem is of significant fundamental value by itself. It is related to the basic understanding of dynamical systems as mappings between input/output functions. To be physical, a forward system, which maps a given input to an output, has to be causal. However, the inverse system, which maps a given output to an input, is not a physical system and does not have to be causal. This is correctly reflected in the stable inversion theory, but the classical inversion theory insists that the inverse system should also be causal. Furthermore, the inversion algorithms from this study may find wide applications in other branches of science and engineering, such as non-destructive evaluation, medical equipment, heat transfer, and others. After all, the inverse problem is a fundamental generic problem in science and engineering.

The remainder of the paper is organized as follows. In the next section we define the class of systems and the class of reference trajectories under consideration and state the problem of stable inversion. Section 3 contains the main results that establish a sufficient invertibility condition and the equivalence of stable inversion to a two-point boundary value problem of reduced-order ordinary differential equations. In §4, the stable inversion technique is applied to the output tracking control of a fourth-order nonlinear non-minimum phase system and is compared with the nonlinear regulation approach in terms of tracking performance. Simulation results are very favourable to our approach and demonstrate the value of stable inverses for dead-beat output tracking control.

2. Framework and problem statement

We consider a nonlinear system of the form

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x) \tag{2}$$

defined on a neighbourhood X of the origin of $\mathbf{R}^n$, with input $u \in \mathbf{R}^m$ and output $y \in \mathbf{R}^p$.

Assumption A1: *$f(x)$, $g_i(x)$ (the ith column of $g(x)$), $i = 1, 2, \ldots, m$, are smooth vector fields and $h_i(x)$, $i = 1, 2, \ldots, p$, are smooth functions on X with $f(0) = 0$ and $h(0) = 0$.*

The system in (1) and (2) may represent a nominal model of a dynamical system. The true system may be subjected to uncertainties such as unmodelled dynamics, modelling errors, etc. These will be considered during the stabilization stage and hence will not be included in this study. In this sense, the assumptions made in this paper are not restrictive since the stable inversion problem is posed based on the nominal model.

The class of reference trajectories (or desired output trajectories) considered reflects practical considerations. For example, all practical signals have a finite horizon, or are defined over a finite interval of time. In trajectory planning, the reference signals are usually defined by interpolating through some pre-calculated points. In such cases, it is not practical to try to use an exosystem to generate the reference signal. Hence, the following assumption is made.

Assumption A2: *The reference output trajectory $y_d(t)$ is a sufficiently smooth function of time satisfying $y_d(t) \equiv 0\ \forall\, t \leqslant t_0$ and $\forall\, t \geqslant t_f$ where $t_f > t_0$ are finite.*

Here, 'sufficiently smooth' means that the signal has continuous derivatives of any order required in the calculation of the stable inverse. Notice that the assumption requires $y_d(t)$ to have a compact support $[t_0, t_f]$. This assumption covers a large family of practical reference trajectories. Furthermore, the development in this paper can be extended with little effort to cover signals whose certain derivatives have a compact support.

In the context of the above, pose the following problem.

Stable inversion problem: Given a reference trajectory $y_d(t)$ satisfying Assumption A2, find an input function $u_d(t)$ and a state trajectory $x_d(t)$ such that

(1) u_d and x_d solve the differential equation

$$\dot{x}_d(t) = f(x_d(t)) + g(x_d(t))\, u_d(t)$$

(2) x_d maps exactly into the reference trajectory

$$h(x_d(t)) = y_d(t), \quad \text{for all } t$$

(3) u_d and x_d are bounded and

$$u_d(t) \to 0, \quad x_d(t) \to 0 \text{ as } t \to \pm\infty \qquad \square$$

If this problem has a unique solution, then the system in (1) and (2) is stably invertible. We call x_d the desired state trajectory and u_d the nominal control input. These will be solved offline first and then incorporated into a dead-beat controller by using the nominal control input as a feed-forward signal and $x - x_d$ as an error signal for feedback.

It is worthwhile discussing the difference between stable inversion defined above and causal inversion proposed by Hirschorn. In the causal inversion, a fixed initial condition (usually zero) is adopted for the forward system as well as for the inverse system. The stability of the inverse solution is then dictated by the stability of the inverse system (or the zero dynamics). In contrast, the stable inversion problem imposes a requirement on the stability of the inverse solution. However, the initial condition is let free (which will be set by the inversion process automatically). This is a reasonable thing to do because the inverse problem is posed on a nominal model of the system and is to be solved offline. Any initial condition error will be taken care of by the feedback stabilizing controller for the forward system.

In solving for the nominal trajectories x_d and u_d the concepts of stable and unstable manifolds (see e.g. Wiggins 1990) of an equilibrium point arise naturally. For the sake of completeness we review the definitions here. Let $z = 0$ be an equilibrium point of an autonomous system defined in an open neighbourhood U of the origin of $\mathbf{R}^n$

$$\dot{z} = f(z)$$

and $\phi_t(z)$ be the flow passing through z at $t = 0$. We define the (local) stable and unstable manifolds W^s, W^u as follows

$$W^s = \{z \in U \mid \phi_t(z) \in U \,\forall\, t \geqslant 0,\ \phi_t(z) \to 0, \text{ as } t \to \infty\}$$
$$W^u = \{z \in U \mid \phi_t(z) \in U \,\forall\, t \leqslant 0,\ \phi_t(z) \to 0, \text{ as } t \to -\infty\}$$

In words, any point on the stable manifold will eventually converge to the equilibrium point in forward time (or as time increases), and any point on the unstable manifold will eventually converge to the equilibrium point in backward time.

The equilibrium point $z = 0$ is said to be hyperbolic if the jacobian matrix Df of f at $z = 0$ has no eigenvalues on the $j\omega$ axis. Let n^s denote the number of eigenvalues of Df in the open left half complex plane, and n^u the number in the open right half-plane. Stable and unstable manifolds W^s and W^u exist locally in the neighbourhood of a hyperbolic fixed point and have dimensions n^s and n^u respectively and $n^s + n^u = n$ where n is of the order of the system.

For convenience, we will use the following notation. Let $\mathbf{N} \triangleq \{0, 1, 2, \ldots\}$, $r = (r_1, r_2, \ldots, r_m)^{\mathrm{T}} \in \mathbf{N}^m$ and $y = [y_1(t), y_2(t), \ldots, y_m(t)]^{\mathrm{T}}$, $t \in \mathbf{R}$. Then we define $|r| \triangleq r_1 + r_2 + \cdots + r_m$ and write

$$y^{(r)} \triangleq \left(\frac{\mathrm{d}^{r_1} y_1}{\mathrm{d}t^{r_1}}, \frac{\mathrm{d}^{r_2} y_2}{\mathrm{d}t^{r_2}}, \ldots, \frac{\mathrm{d}^{r_m} y_m}{\mathrm{d}t^{r_m}}\right)^{\mathrm{T}}$$

We will use the bold number **1** to denote the vector $(1, 1, \ldots, 1)^{\mathrm{T}}$ so that

$$y^{(\mathbf{1})} = \dot{y} = \left(\frac{\mathrm{d}y_1}{\mathrm{d}t}, \frac{\mathrm{d}y_2}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}y_m}{\mathrm{d}t}\right)^{\mathrm{T}}$$

If y: $\mathbf{R}^n \to \mathbf{R}^m$ and f: $\mathbf{R}^n \to \mathbf{R}^n$ and $g = (g_1, g_2, \ldots, g_k)$ with g_i: $\mathbf{R}^n \to \mathbf{R}^n$, we define

$$L_f^r y \triangleq (L_f^{r_1} y_1, L_f^{r_2} y_2, \ldots, L_f^{r_m} y_m)^{\mathrm{T}}$$

$$L_g y \triangleq (L_{g_1} y, L_{g_2} y, \ldots, L_{g_k} y)$$

3. Stable inversion of non-minimum phase systems

Consider a nonlinear system of the form (1) and (2) with the same number m of inputs as outputs and

$$y = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$$
$$u = (u_1, u_2, \ldots, u_m)^{\mathrm{T}}$$
$$h(x) = [h_1(x), h_2(x), \ldots, h_m(x)]^{\mathrm{T}}$$
$$g(x) = [g_1(x), g_2(x), \ldots, g_m(x)]$$

Assumption A3: *The system in* (1) *and* (2) *has well-defined relative degree* $r = (r_1, r_2, \ldots, r_m)^{\mathrm{T}} \in \mathbf{N}^m$ ($|r| \leqslant n$) *at the equilibrium point* 0, *i.e. in an open neighbourhood of* 0,

(i) *for all* $1 \leqslant j \leqslant m$, *for all* $1 \leqslant i \leqslant m$, *for all* $k < r_i - 1$ *and* $k \geqslant 0$, *and for all* x

$$L_{g_j} L_f^k h_i(x) = 0$$

(ii) *the* $m \times m$ *matrix*

$$\beta(x) \triangleq L_g L_f^{r-\mathbf{1}} h(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \ldots & L_{g_m} L_f^{r_1-1} h_1(x) \\ L_{g_1} L_f^{r_2-1} h_2(x) & \ldots & L_{g_m} L_f^{r_2-1} h_2(x) \\ \ldots & \ldots & \ldots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \ldots & L_{g_m} L_f^{r_m-1} h_m(x) \end{bmatrix}$$

is non-singular.

In words, this assumption means the following. If we take derivatives of the output, the first time any input will appear is in the r_ith derivative of y_i and at this time the coefficient matrix of the input is non-singular. Note that since the control u does not appear explicitly in (2), we have $r_i \geqslant 1$ for all i. Therefore, $r - \mathbf{1} \in \mathbf{N}^m$ and the operation in the definition of β is well defined.

Under this assumption, the system can be partially linearized. To do this, we differentiate y_i until at least one u_j appears explicitly. This will happen at exactly the r_ith derivative of y_i due to Assumption A3. Define $\xi_k^i = y_i^{(k-1)}$ for $i = 1, 2, \ldots, m$ and $k = 1, \ldots, r_i$, and denote

$$\begin{aligned}\xi &= (\xi_1^1, \xi_2^1, \ldots, \xi_{r_1}^1, \xi_1^2, \ldots, \xi_{r_2}^2, \ldots, \xi_{r_m}^m)^{\mathrm{T}} \\ &= (y_1, \dot{y}_1, \ldots, y_1^{(r_1-1)}, y_2, \ldots, y_2^{(r_2-1)}, \ldots, y_m^{(r_m-1)})^{\mathrm{T}}\end{aligned}$$

In the trivial case when $|r| = n$, there will be no zero dynamics and the inversion problem becomes a kinematic or algebraic inversion. Hence, it is assumed that $|r| < n$ strictly. Choose η, an $n-|r|$ dimensional function on $\mathbf{R}^n$, such that $(\xi^{\mathrm{T}}, \eta^{\mathrm{T}})^{\mathrm{T}} = \Psi(x)$ forms a change of coordinates with $\Psi(0) = 0$ (see for example Isidori 1989). Notice that η can be selected (which is usually more difficult) such that the system will be transformed into its normal form. However, the normal form is not required in our derivation and η is allowed to be any functions that lead to a change of coordinates. In the new coordinates, the system dynamics of (1) become

$$\begin{cases} \dot{\xi}_1^i = \xi_2^i \\ \quad\vdots \\ \dot{\xi}_{r_i-1}^i = \xi_{r_i}^i & \text{for } i = 1, \ldots, m \\ \dot{\xi}_{r_i}^i = \alpha_i(\xi, \eta) + \beta_i(\xi, \eta)\, u \end{cases}$$
$$\dot{\eta} = q_1(\xi, \eta) + q_2(\xi, \eta)\, u$$

which, in a more compact form, is equivalent to

$$y^{(r)} = \alpha(\xi, \eta) + \beta(\xi, \eta)\, u \tag{3}$$
$$\dot{\eta} = q_1(\xi, \eta) + q_2(\xi, \eta) u \tag{4}$$

where

$$\begin{aligned}\alpha(\xi, \eta) &= L_f^r h(\Psi^{-1}(\xi, \eta)) \\ \beta(\xi, \eta) &= L_g L_f^{r-1} h(\Psi^{-1}(\xi, \eta))\end{aligned}$$

$\alpha(0, 0) = 0$ since $f(0) = 0$, and α_i and β_i are the ith row of α and β respectively. In order to solve the inversion problem, we must set $y(t) \equiv y_d(t)\ \forall\, t$ due to requirement 2 of the problem statement. Then immediately we have $y^{(r)}(t) \equiv y_d^{(r)}(t)$ and

$$\xi = \xi_d \triangleq (y_{d1}, \dot{y}_{d1}, \ldots, y_{d1}^{(r_1-1)}, y_{d2}, \ldots, y_{d2}^{(r_2-1)}, \ldots, y_{dm}^{(r_m-1)})^{\mathrm{T}}$$

In the inverse system, the reference trajectory and its derivatives will be the input and the original input u will be the new output. Since by the relative degree assumption, $\beta(\xi, \eta)$ is non-singular, then (3) can be used to solve for the new output map for the inverse system

$$u \triangleq [\beta(\xi_d, \eta)]^{-1}[y_d^{(r)}(t) - \alpha(\xi_d, \eta)] \tag{5}$$

With this, (4) becomes what we call the reference dynamics, or the zero dynamics driven by the reference output trajectory

$$\dot{\eta} = p(y_d^{(r)}, \xi_d, \eta) \tag{6}$$

where

$$p(y_d^{(r)}, \xi_d, \eta) \triangleq q_1(\xi_d, \eta) + q_2(\xi_d, \eta)\,[\beta(\xi_d, \eta)]^{-1}[y_d^{(r)} - \alpha(\xi_d, \eta)] \tag{7}$$

When $y_d^{(r)} = 0$ and $\xi_d = 0$, equation (6) represents the zero dynamics of the system in (1) and (2). It is clear now that an integration of the reference dynamics defines a trajectory of the original state through the inverse coordinate transformation

$$x = \Psi^{-1}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

and an input trajectory by (5). Now the question is how to integrate the reference dynamics to generate a bounded input solving the stable inversion problem, since the reference dynamics may be unstable in both positive and negative time directions, in general.

Assumption A4: *$Dp(0,0,0)$ has no eigenvalues on the imaginary axis.*

In other words, this assumption requires that the zero dynamics have a hyperbolic fixed point at the origin. Since the requirement is imposed on the nominal system, this assumption is quite reasonable. Furthermore, small changes can always be made to the nominal system so that it satisfies Assumption A4 and the resulting modelling error will be left to the feedback stabilizing controller.

For reference trajectories with compact support, the reference dynamics become the autonomous zero dynamics for t outside the compact interval $[t_0, t_f]$. Under Assumption A4, $\eta = 0$ is a hyperbolic equilibrium point of the autonomous zero dynamics $\dot\eta = p(0,0,\eta)$. Hence there exist stable and unstable manifolds W^s and W^u. Locally W^u can be defined by an equation $B^u(\eta) = 0$ and, similarly, W^s can be defined by $B^s(\eta) = 0$. The following theorem is then in order.

Theorem 1: *Under Assumptions* A1–A4, *the stable inversion problem has a solution if and only if the following two-point boundary value problem has a solution*

$$\dot\eta = p(y_d, \xi_d, \eta)$$

subject to

$$\left.\begin{aligned} B^u(\eta(t_0)) &= 0 \\ B^s(\eta)(t_f)) &= 0 \end{aligned}\right\} \tag{8}$$

Proof:

Necessity. Suppose $x_d(t)$ and $u_d(t)$ solve the stable inversion problem. Then $x_d(t)$ and $u_d(t)$ have to satisfy the differential equation (1). Let $(\xi^T, \eta^T)^T = \Psi(x_d)$. Then ξ and η satisfy equation (1) or equivalently (3) and (4) with u substituted by u_d. Besides, since $y = h(x_d) = y_d$ by assumption, $\xi = \xi_d$ and $y^{(r)} = y_d^{(r)}$. Therefore by (3)

$$y_d^{(r)} = \alpha(\xi_d, \eta) + \beta(\xi_d, \eta)\, u_d$$

which yields

$$u_d = [\beta(\xi_d, \eta)]^{-1}(y_d^{(r)} - \alpha(\xi_d, \eta))$$

Substituting this into the η dynamics of (4) and comparing the resulting right-hand side with the definition of $p(y_d, \xi_d, \eta)$ in (7), we recognize that η satisfies equation (6).

Now we only need to show that η also satisfies the boundary condition. This is easy. Since by assumption, $x_d(t) \to 0$ as $t \to -\infty$ and $\Psi(0) = 0$, thus $\eta(t) \to 0$ as $t \to -\infty$ also. By definition of the unstable manifold, $\eta(t) \in W^u$ for all $t \leqslant t_0$. Therefore, $B^u(\eta(t_0)) = 0$. Similar arguments show that $B^s(\eta(t_f)) = 0$.

Sufficiency. Suppose η_d solves the above two-point boundary value problem. Then η_d is bounded, and $\eta_d(t) \to 0$ as $t \to \infty$ since $\eta_d(t_f) \in W^s$ and $\eta_d(t) \to 0$ as $t \to -\infty$ since $\eta_d(t_0) \in W^u$. Also, $\xi_d = 0$ for $t \leqslant t_0$ and for $t \geqslant t_f$.

Let

$$x_d = \Psi^{-1}(\xi_d, \eta_d)$$
$$u_d = [\beta(\xi_d, \eta_d)]^{-1}(y_d^{(r)} - \alpha(\xi_d, \eta_d))$$

Then, x_d and u_d are bounded, and $x_d(t), u_d(t) \to 0$ as $t \to \infty$ or $t \to -\infty$, since $\Psi^{-1}(0,0) = 0$ and $\alpha(0,0) = 0$. By the definition of ξ, $y = y_d$. This completes the proof. □

Theorem 2: *Under Assumptions A1–A4, the two-point boundary value problem of equations (6) and (8) has a unique solution provided* $\|\bar{\xi}_d\|_\infty \triangleq \sup_{t\in[t_0,t_f]} \|\bar{\xi}_d(t)\|_2$ *is sufficiently small, where* $\bar{\xi}_d \triangleq ((y^{(r)})^T, \xi_d^T)^T$.

Proof: First, due to the smoothness of f, g, h and y_d, the right-hand side of (8), $p(\bar{\xi}_d, \eta)$ is smooth in η and t. Therefore, for any small initial condition η and small $\bar{\xi}_d$, there exists a unique solution to equation (6) (see Miller and Michel 1982). Let $\phi^{\bar{\xi}_d}_{t_0,t}(\eta)$ be the flow starting from η at t_0 driven by $\bar{\xi}_d$. Then it remains to show that there exists a unique point η on the unstable submanifold that will be mapped to a point on the stable submanifold at t_f by $\phi^{\bar{\xi}_d}_{t_0,t_f}(\eta)$. Or, equivalently

$$F(\eta, \bar{\xi}_d) \triangleq \begin{bmatrix} B^u(\phi^{\bar{\xi}_d}_{t_0,t_f}(\eta)) \\ B^s(\eta) \end{bmatrix} = 0$$

has a unique solution for each $\bar{\xi}_d$ sufficiently small. This can be established using the implicit function theorem. To do this, we need F to be continuously differentiable and $D_\eta F(0,0)$ to be invertible. Since the stable and unstable submanifolds are smooth and the flow $\phi^{\bar{\xi}_d}_{t_0,t_f}(\eta)$ is continuously differentiable in $\bar{\xi}_d$ and η (see Miller and Michel 1982), therefore, F is continuously differentiable. In addition, since the two submanifolds intersect transversally at the equilibrium point, we have that $D_\eta F(0,0)$ is an isomorphism. Therefore, the implicit function theorem guarantees a unique solution η for each $\bar{\xi}_d$ and the dependence is continuously differentiable. The flow starting from this η solves the two-point boundary value problem. □

A geometric interpretation of the $x_d(t)$ evolution is shown in Fig. 1. For clarity we have shown a case of output slewing so that the internal dynamics manifolds for $h(x) = y_f = y_d(t_f)$ and $h(x) = y_0 = y_d(t_0)$ are separate. For y_d satisfying Assumption 2, these two manifolds coincide. The non-causal part of the nominal controls drives the internal states of the system along the unstable manifold of the initial internal dynamics manifold to a particular initial condition $x_d(t_0)$ while maintaining the desired constant system output $y = y_0 = y_d(t_0)$. This initial condition guarantees two things: (1) the desired reference output trajectory is easily reproduced with bounded input and states; (2) the internal states land on the stable manifold of the final internal dynamics manifold at the end of output tracking. With this nice final condition, the internal states will converge to zero along the stable manifold without affecting the output.

Here we see that the stable inversion problem is transformed into a two-point boundary value problem for which the number of equations is reduced. However, it is still a non-trivial numerical problem. The difficulty arises because of the instability of the reference dynamics in both positive and negative time. Existing approaches, for example the shooting method, do not perform well numerically for unstable systems.

In the case of minimum-phase systems, the reference dynamics are asymptotically stable in the forward time. The size of the stable manifold is the same as that of the zero dynamics manifold and the unstable manifold reduces to the origin only. Therefore, the boundary condition $B^u(\eta(t_0)) = 0$, reduces to $\eta(t_0) = 0$ and $B^s(\eta(t_f)) = 0$ imposes no

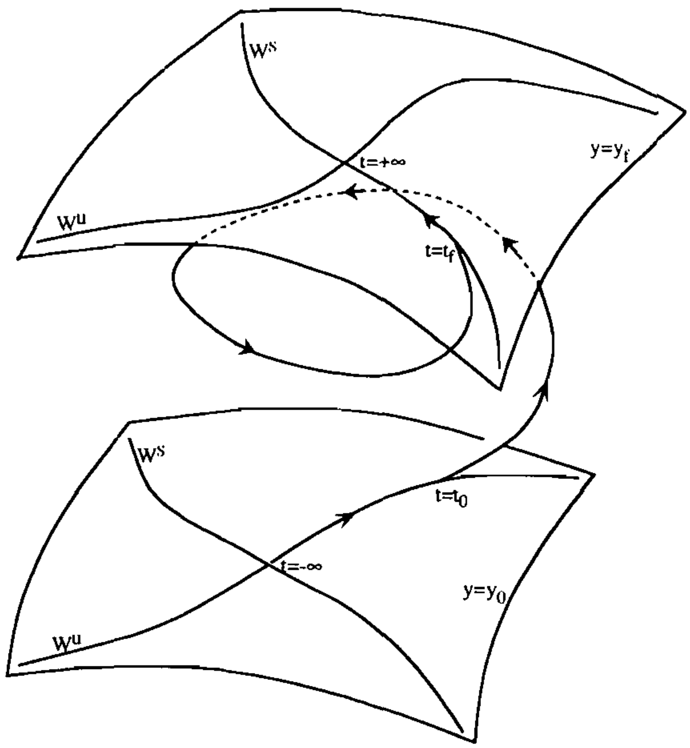

Figure 1. Geometric view of stable inverse.

extra constraints. Hence the two-point boundary value problem reduces to a simple initial value problem with asymptotically stable dynamics, and can be easily integrated in the forward time. This is Hirschorn's approach. Similarly, if the zero dynamics are completely unstable, the two-point boundary value problem reduces to a final-value problem and can be easily integrated in backward time.

Another simple situation is when the stable and unstable parts of the reference dynamics can be decoupled by a change of coordinates. This happens when the reference dynamics are a linear time-invariant system driven by the reference output and its derivatives. In such cases, we can easily integrate the stable part in forward time and the unstable part in backward time.

4. Application to output tracking

The stable inversion theory is expected to play a key role in achieving high precision output tracking control for nonlinear non-minimum phase systems. To achieve this, the stable inversion problem is first solved offline to obtain a nominal control input u_d and a desired state trajectory x_d that maps into the reference output trajectory. Then the nominal control u_d is used as a feed-forward, and a feedback controller is used to stabilize the desired state trajectory x_d. However, since the control part is not the focus of this paper, here we use a specific example to demonstrate the value of stable inverses in output tracking. The nonlinear regulation is also applied to the same example and the tracking performances by the two approaches are compared with each other.

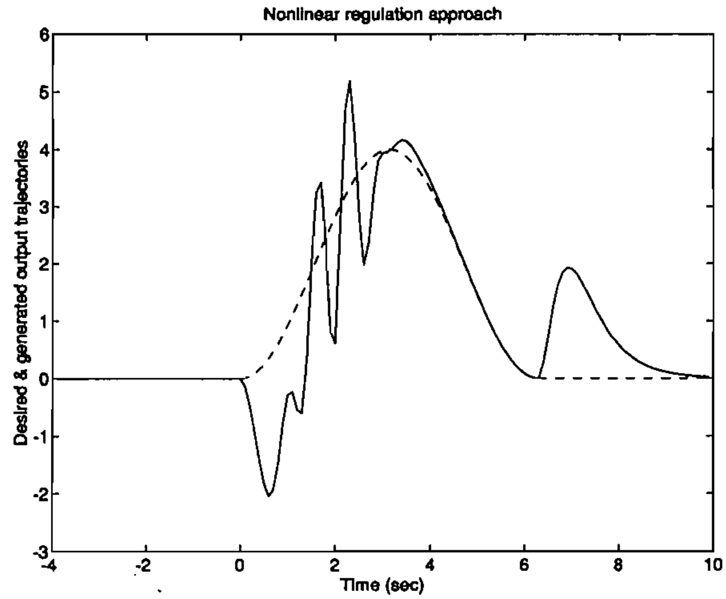

Figure 2. Desired and actual output in nonlinear regulation approach.

The example system is selected such that both time integration of the reference dynamics and the solution to the nonlinear partial differential equations are manageable. The system is a slightly nonlinear single-input single-output system described by the following equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -x_1+x_2 \\ -3x_2+x_1^3 \\ x_1-2x_3 \\ -x_4+x_3^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2+\sin^2 x_4 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = x_1 - 3x_3$$

The reference output trajectory is given by

$$y_d = \begin{cases} 2(1-\cos(t)) & t \in [0, \pi] \\ 0 & \text{otherwise} \end{cases}$$

and is depicted in Fig. 2 with a dashed curve.

First, let us consider the regulator approach. The reference signal can be exactly generated by the following linear time-invariant exosystem

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1 \\ \dot{w}_3 &= 0 \\ y_d &= w_3 - w_1 \end{aligned}$$

The initial conditions are set and reset as follows

$$\begin{aligned} w_1(-\infty) = w_2(-\infty) = w_3(-\infty) &= 0 \\ w_1(0) = w_3(0) = 2, \quad w_2(0) &= 0 \\ w_1(2\pi = w_2(2\pi) = w_3(2\pi) &= 0 \end{aligned}$$

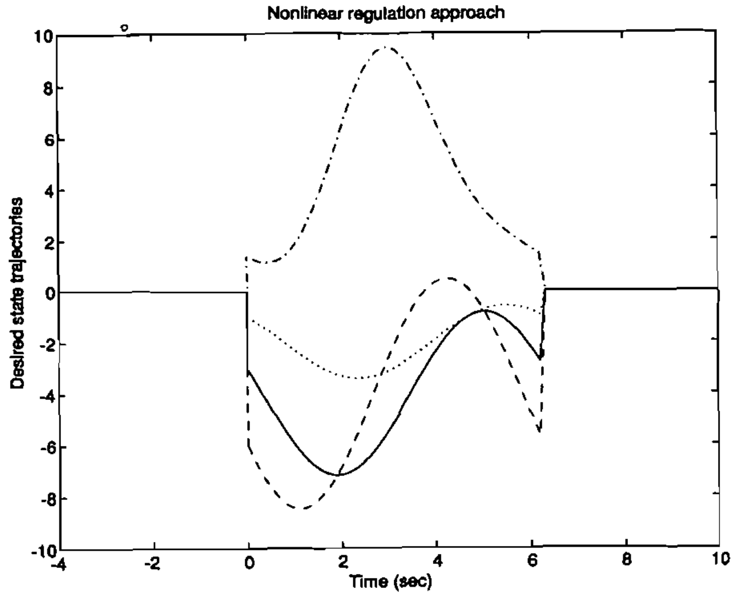

Figure 3. Desired state trajectories in nonlinear regulation approach.

The zero error manifold, $x = x(w)$ and $u = u(w)$, is obtained by solving a system of nonlinear partial differential equations, which is, in general, extremely difficult if not impossible. For this example, the partial differential equations are as follows

$$\frac{\partial x_1(w)}{\partial w_1} w_2 - \frac{\partial w_1(w)}{\partial w_2} w_1 = -x_1(w) + x_2(w)$$

$$\frac{\partial x_2(w)}{\partial w_1} w_2 - \frac{\partial w_2(w)}{\partial w_2} w_1 = -3x_2(w) + x_1^3(w) + (2 + \sin^2 x_4(w))\, u(w)$$

$$\frac{\partial x_3(w)}{\partial w_1} w_2 - \frac{\partial x_3(w)}{\partial w_2} w_1 = x_1(w) - 2x_3(w)$$

$$\frac{\partial x_4(w)}{\partial w_1} w_2 - \frac{\partial w_4(w)}{\partial w_2} w_1 = -x_4(w) + x_3^2(w)$$

subject to

$$x_1(w) - 3x_3(w) = w_3 - w_1$$

Fortunately, we are able to solve this system in closed form. Details will be omitted here and the final solutions are

$$x_1(w) = -\frac{5}{2} w_1 - \frac{3}{2} w_2 - 2w_3$$

$$x_2(w) = -7w_1 - 4w_2 - 2w_3$$

$$x_3(w) = -\frac{1}{2}(w_1 + w_2) - w_3$$

$$x_4(w) = \frac{7}{20} w_1^2 + \frac{3}{20} w_2^2 + \frac{1}{10} w_1 w_2 - w_1 w_3 + w_3^2$$

$$u(w) = (-17w_1 - 7w_2 + 18w_3)/(2 + \sin^2 x_4(w))$$

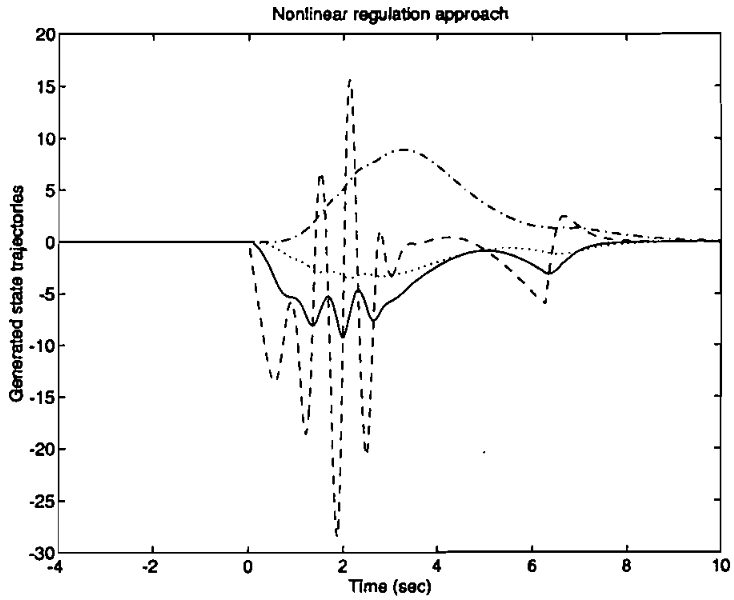

Figure 4. Actual state trajectories in nonlinear regulation approach.

Now note that with $u = 0$, the forward system is locally asymptotically stable since the jacobian matrix of $f(x)$ at $x = 0$ is clearly

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and has all its eigenvalues negative. Therefore, for simplicity, we can choose the feedback gain to be zero.

This completes the regulator design. The simulation results are shown in Figs 2 to 5. Figure 2 compares the desired and actual output trajectory; Fig. 3 shows the state trajectories solving the partial differential equations, Fig. 4 the actual state trajectories in forward simulation, and Fig. 5 the control input applied. Note that the output generated by the regulator does asymptotically track the reference trajectory as predicted by theory. This is shown by the segments from $t = 4$ to $t = 2\pi$ and $t > 9$. However, there are substantial transient tracking errors right after the beginning of manoeuvre at $t_0 = 0$ and right after the end of manoeuvre at $t_f = 2\pi$. This phenomenon is not a special case of this example, but rather general.

Next, let us consider the stable inversion approach. To linearize the system partially, we differentiate the output y to yield

$$\dot{y} = \dot{x}_1 - 3\dot{x}_3 = -4x_1 + x_2 + 6x_3$$

Since the control u does not appear explicitly, we differentiate $\dot{y}$ again to yield

$$\ddot{y} = -4(-x_1 + x_2) - 3x_2 + x_1^3 + (2 + \sin^2 x_4)u + 6(x_1 - 2x_3)$$

$$= (10x_1 - 7x_2 - 12x_3 + x_1^3) + (2 + \sin^2 x_4)u \triangleq \alpha(x) + \beta(x)u$$

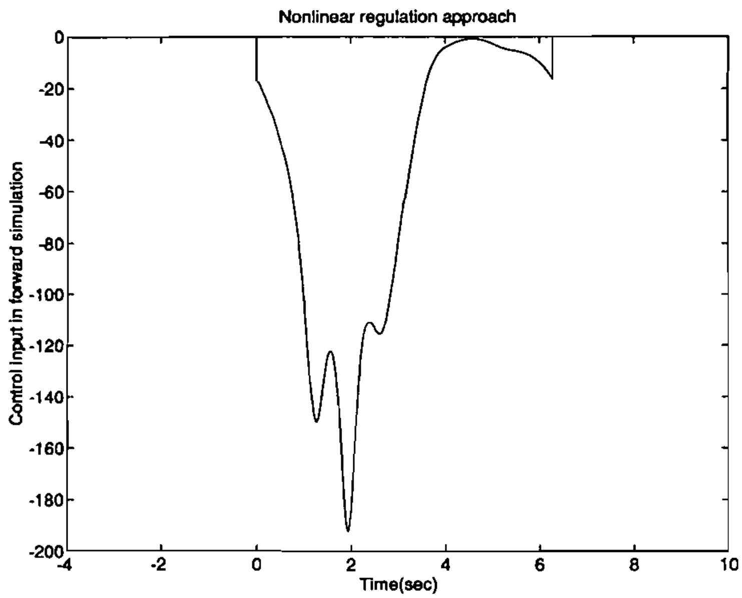

Figure 5. Control input in nonlinear regulation approach.

Now not only does u appear, but also its coefficient $\beta(x) \neq 0$ for all x. Hence, we can set

$$u(x) = \frac{1}{\beta(x)}(\ddot{y}_d - \alpha(x)) \tag{9}$$

and introduce a change of coordinates

$$\begin{bmatrix} y \\ \dot{y} \\ \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 & 0 \\ -4 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The inverse transformation is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 4 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \eta_1 \\ \eta_2 \end{bmatrix} \tag{10}$$

Using (9), the system in the new coordinates becomes $\ddot{y} = \ddot{y}_d$ and

$$\begin{aligned} \dot{\eta}_1 &= \eta_1 + y \\ \dot{\eta}_2 &= -\eta_2 + \eta_1^2 \end{aligned} \tag{11}$$

By setting $y = y_d$ in the above, we obtain the reference dynamics. For t outside $[t_0, t_f] = [0, 2\pi]$, $y_d = 0$, we have the zero dynamics

$$\begin{aligned} \dot{\eta}_1 &= \eta_1 \\ \dot{\eta}_2 &= -\eta_2 + \eta_1^2 \end{aligned}$$

It is an autonomous system with a hyperbolic fixed point at 0, since the jacobian matrix

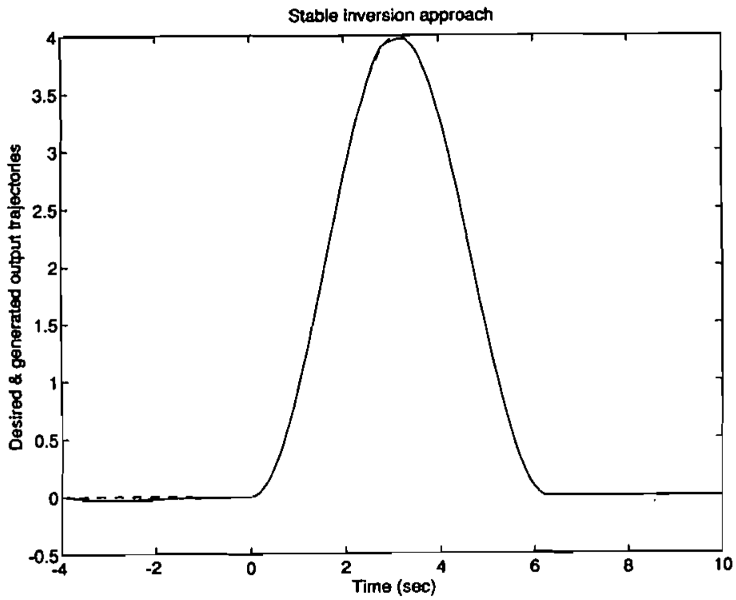

Figure 6. Desired and actual output in stable inversion approach.

has eigenvalues 1 and -1. Therefore, there exist stable and unstable manifolds. The stable manifold can be easily seen to be characterized by

$$\eta_1 = 0$$

and the unstable manifold is characterized by

$$\eta_2 = \frac{\eta_1^2}{3} \tag{12}$$

Therefore, the two-point boundary value problem is given by

$$\dot{\eta}_1 = \eta_1 + y_d, \quad \eta_1(t_f) = 0,$$

$$\dot{\eta}_2 = -\eta_2 + \eta_1^2, \quad \eta_2(t_0) = \frac{\eta_1^2(t_0)}{3}$$

This particular example is in a triangular form and can be easily solved. The first equation is antistable with a final value condition, it can be easily integrated backward in time to obtain η_1. The integration is continued into the time $t < t_0$ and stopped when $|\eta_1|$ is sufficiently small. Once we have η_1, the second equation is a stable system with an initial condition and is driven by a known input η_1^2. Integration forward in time is no problem either. For the part of η_2 before t_0, we use the simple algebraic relation of equation (12) since the trajectory remains on the unstable manifold.

Once η_1 and η_2 are calculated, the desired trajectory of the original states can be obtained using the inverse coordinate transformation in (10) with $y = y_d$ and $\dot{y} = \dot{y}_d$. Then the nominal control input is calculated according to the linearizing feedback law in (9). Note that the η_i's are non-zero for $t < t_0$, the u thus obtained is also non-zero for $t < t_0$, corresponding to a non-causal input.

Again, since the forward system is asymptotically stable, for simplicity we choose zero feedback gain as the stabilizing controller for the forward system. The simulation results are shown in Figs 6 to 9. Figure 6 compares the reference trajectory and output

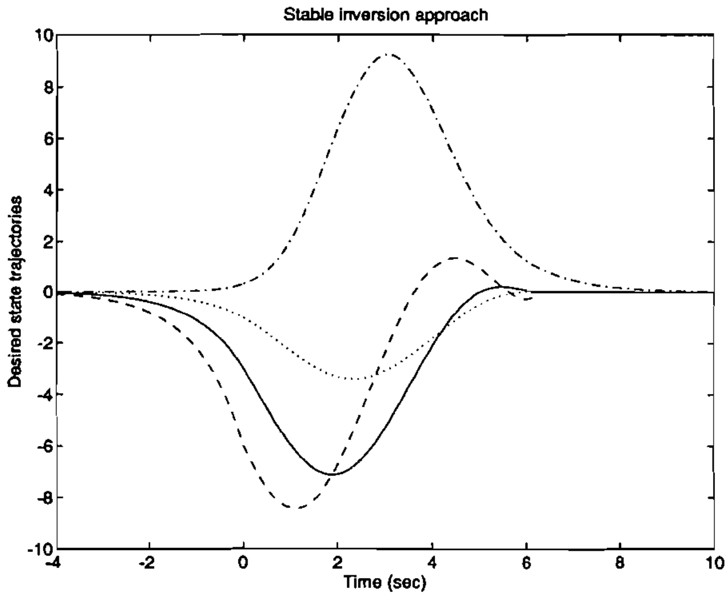

Figure 7. Desired state trajectories in stable inversion approach.

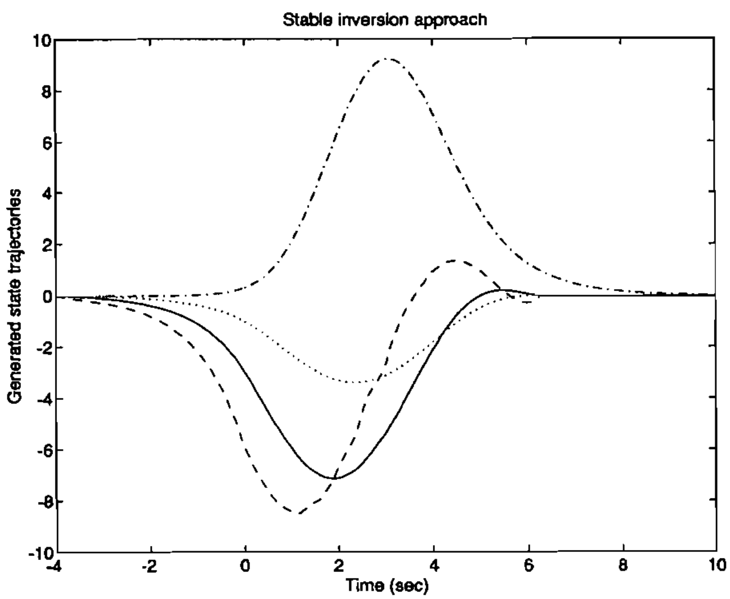

Figure 8. Actual state trajectories in stable inversion approach.

trajectory generated by the stable inversion approach; Fig. 7 shows the desired state trajectories generated offline by inversion, Fig. 8 the actual state trajectories generated by the truncated nominal input, and Fig. 9 the control input applied to the forward system.

The following observations are in order. First, almost perfect output tracking is obtained using a mild control effort. A well designed stabilizing feedback controller is expected to improve these small errors due to approximation by truncation. Second, internal stability is not jeopardized by the exact output tracking as is always the case

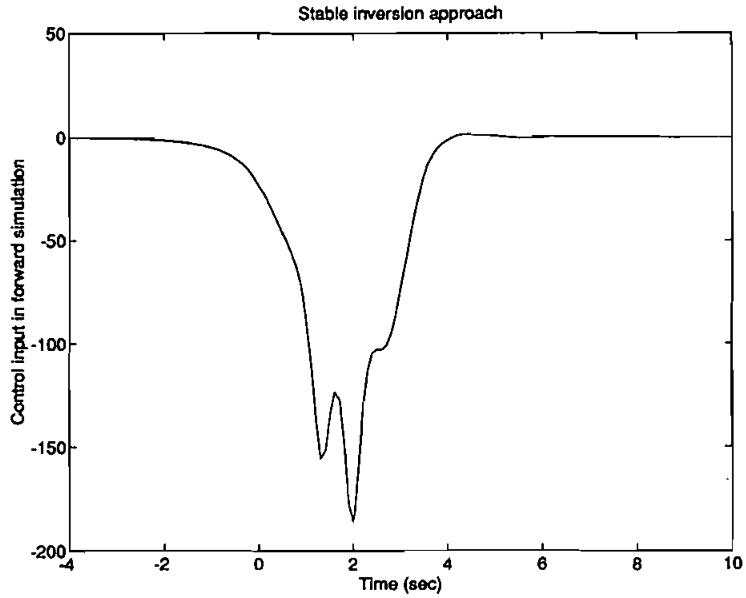

Figure 9. Control input in stable inversion approach.

in classical inversion for non-minimum phase systems. Third, the control input to the forward system is non-zero for $t \leqslant 0$, during which time the reference output is identically zero. This 'non-causal' effect is the price to pay for exact output tracking with stability. Finally, the desired state trajectories and the ones generated by forward simulation are almost identical. Thus, the desired state trajectories calculated by stable inversion correspond to physically possible trajectories that can be precisely achieved. On the contrary, the desired state trajectories calculated in the nonlinear regulation approach do not correspond to physically achievable trajectories. The difference comes from the fact that the $x_d(t)$ in stable inversion solve the differential equation of the forward system, while the $x_d(t)$ in nonlinear regulation are the solutions of a set of partial differential equations and do not satisfy the forward dynamics.

5. Conclusions

In this paper, we have introduced the notion of stable inversion of non-minimum phase nonlinear systems. The solution of the stable inversion problem is shown to be equivalent to a two-point boundary value problem of the zero dynamics driven by reference output. A condition for stable invertibility has been established for a class of nonlinear systems with well-defined relative degree and hyperbolic zero dynamics. These results, in conjunction with Hirschorn's, show that there are multiple inverses for non-minimum phase systems: bounded but non-causal solutions produced with our method and causal but unbounded solutions produced using Hirschorn's techniques. These inversion techniques are fundamental to nonlinear tracking controllers that use feed-forward produced by inversion in conjunction with stabilizing feedback. The application results included in this paper have demonstrated the value of stable inversion in achieving high-precision stable output tracking. Future work will include properties of stable inversion and synthesis of stabilizing feedback controllers for output tracking. Applications to practical engineering problems will also be studied.

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