Proceedings of the 35th
Conference on Decision and Control
Kobe, Japan • December 1996

Asymptotic Feedback Stabilization of a Nonholonomic Mobile Robot using a Nonlinear Oscillator

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Abstract

This paper addresses the problem of point-to-point stabilization of a two-wheeled mobile robot. It is well known that there does not exist a smooth static state feedback controller for the stabilization of the mobile robot to arbitarary fixed postures. Researchers in the past have therefore used smooth time-varying control, piecewise smooth control and hybrid control for stabilization. In this paper we present a piecewise smooth dynamic controller for the global asymptotic stabilization of the mobile robot. As different from other piecewise smooth controllers, our controller has at most one switching, otherwise it is smooth. Our controller is also dynamic in the sense that the control inputs are obtained as a solution to a first order differential equation and convergence to the desired posture is guaranteed for any nonzero initial condition. The controller, inherently simple in its formulation, uses the dynamics of a nonlinear oscillator that plays a key role in preventing the mobile robot from getting stuck at any point other than the desired posture. The controller guarantees the simultaneous asymptotic stabilization of the states of the mobile robot to their desired values and the states of the oscillator to zero. Simulation results presented aptly demonstrate the efficacy of the stabilizing control.

1. Introduction

The problems of motion planning and feedback stabilization of mobile robots have been studied in recent years with the aim of developing autonomous wheeled vehicles. Researchers have looked at the motion planning problem in search for feasible trajectories and a number of solutions have been proposed - in some of them optimality of the trajectories have been ensured while in others the problem of path planning among obstacles has been addressed. While certain motion planning algorithms have been modified into feedback control strategies, some researchers have studied the feedback control problem separately. Specifically, they have addressed the problems of tracking a geometric path and the problem of point-to-point stabilization.

As a pre-cursor to the motion planning problem, the controllability of the mobile robot was ascertained (Lafferiere and Sussman, 1990; Laumond, 1992, 1993). A solution to the planning problem was subsequently proposed by Murray and Sastry (1991). Here the system was converted into a "chained" form and steered using sinusoidal inputs. This approach can be applied to nonholonomic systems that can be converted into the "chained form" and can be used for the motion planning of the car-like mobile robot with n trailers. The procedure for the conversion of the kinematics of a mobile robot with n trailers into the chained form was developed by Sordalen (1993). The study of shortest paths for a mobile robot in the absence of obstacles was carried out by Dubins (1957), Reeds and Shepp (1990) and by Boissonnat et.al. (1992). The obstacle avoidance problem was addressed by Laumond et.al. (1994).

One of the feedback control problems of a mobile robot can be described as the task of stabilizing the motion of the robot about a time-indexed trajectory or a geometric path determined apriori by a motion planner. This problem has received solutions involving classical nonlinear control techniques (Canudas de Wit and Samson, 1991; Kanayama et.al., 1990; Sampei et.al., 1991; Samson and Ait-Abderrahim, 1990). The problem of tracking a reference trajectory is defined as the task of finding a feedback control law that asymptotically stabilizes the motion of the robot about the reference trajectory provided that the reference vehicle is not at rest at all times. Hence stabilization to a desired posture is not included in the definition of the tracking problem.

The feedback stabilization problem is inherently more difficult than the problems of tracking or path following. This is due to the non-existence of a stabilizing static smooth pure state feedback law for the nonlinear model representing a mobile robot. This negative result is the consequence of a theorem due to Brockett (1983). In recent years a control technique using time-varying state feedback was proposed to achieve smooth point-to-point stabilization. The explicit time-varying nature of the control, first proposed by Samson (1990), does not violate Brockett's theorem (1983). Several studies have been carried out thereafter. General existence results can be found in (Coron, 1992), and explicit time-periodic feedback laws for a class of nonlinear systems can be found in (Pomet, 1992). The convergence properties of time-varying feedback has been studied by Gurvits and Li (1992) and M'Closkey and Murray (1993). In general, the smooth time-varying controllers are known to have slow convergence rates. An alternative method for point-to-point stabilization is to design piecewise-smooth controllers. The work on discontinuous controllers for nonholonomic systems was initiated by Bloch et.al. (1990). A controller for the exponential convergence of a mobile robot to a desired posture using piecewise continuous feedback was developed by Canudas de Wit and Sordalen (1992) and Sordalen (1993). Though piecewise continuous controllers can guarantee faster convergence rates, they may have multiple switchings and may not be always practical. For example, multiple switchings in the controller of a nonholonomic flexible space multibody system may excite the high frequency modes of the system. More recently, hybrid controllers that are both piecewise smooth and time-varying, have been proposed for mobile robots and other nonholonomic systems. A detailed review of different feedback control strategies developed for nonholonomic systems can be found in (Kolmanovsky and McClamroch, 1995).

In this paper we present a hybrid controller, piecewise-smooth and dynamic, for the global asymptotic stabilization of a two-wheeled mobile robot to arbitrary desired postures. The control action of our piecewise smooth controller can be designed to be continuous by proper choice of controller parameters. As different from existing piecewise-smooth controllers, our controller has at most one switching and in the event that there is no switching, as in the case of the parallel parking problem, our controller is smooth. It is also different from existing controllers in the sense that it is a dynamic controller. Research effort in the past have been mainly concentrated on designing explicitly time-varying and piecewise smooth controllers and the possibility of using dynamic controllers for feedback stabilization (Kapitanovsky, et.al., 1993; Tilbury, et.al., 1995) has been largely overlooked. In our dynamic controller, the inputs to the mobile robot are obtained

0-7803-3590-2/96 $5.00 © 1996 IEEE

as a solution to a first-order differential equation and asymptotic stabilization of the robot to the desired posture is guaranteed for any nonzero initial condition. The dynamic controller proposed herein is physically motivated and uses the dynamics of a nonlinear oscillator which plays a key role in preventing the mobile robot from getting stuck at any point other than the desired posture.

The rest of this paper is organized as follows. In section 2 the kinematics of the mobile robot is discussed. In section 3 we design a controller that converges the mobile robot onto the line containing the desired posture. This prepares us for the discussion in section 4 where the problem of feedback stabilization to a desired posture is addressed. In section 4 the stabilizing controller is presented along with its proof. Section 5 provides results of numerical simulation and concluding remarks are included in section 6.

2. Mobile Robot Kinematics

The kinematics of a mobile robot with two driving wheels are given as

$$\begin{aligned} \dot{x} &= \cos\theta\, u_1 \\ \dot{y} &= \sin\theta\, u_1 \\ \dot{\theta} &= u_2 \end{aligned} \tag{1}$$

where, x and y denote the Cartesian coordinates of the robot and θ denotes the orientation of the robot with respect to the positive x axis. u_1 and u_2 are the linear and angular velocities of the robot, they are also the controls of the robot.

In reference to Fig.1, it is assumed that x, y and θ denote the current configuration of the robot and x_d, y_d and θ_d denote the desired configuration of the robot. Let $\mathcal{L}$ denote the straight line passing through the coordinates x_d and y_d with the slope of θ_d and let M denote the projection of the point (x, y) on $\mathcal{L}$. If p denotes the distance between the points M and (x, y), and d denotes the distance between points M and (x_d, y_d), then, p and d can be expressed as follows

$$\begin{aligned} p &= \left(\Delta x^2 + \Delta y^2\right)^{1/2} \sin(\gamma - \theta_d) \\ &= \Delta y \cos\theta_d - \Delta x \sin\theta_d \end{aligned} \tag{2}$$

$$\begin{aligned} d &= \left(\Delta x^2 + \Delta y^2\right)^{1/2} \cos(\gamma - \theta_d) \\ &= \Delta x \cos\theta_d + \Delta y \sin\theta_d \end{aligned} \tag{3}$$

where Δx and Δy are defined as $\Delta x \triangleq (x_d - x)$ and $\Delta y \triangleq (y_d - y)$, respectively. If the expressions in Eqs.(2) and (3) are differentiated, it can be easily shown that

$$\dot{p} = \quad \dot{x}\sin\theta_d - \dot{y}\cos\theta_d = \quad u_1 \sin(\Delta\theta) \tag{4}$$

$$\dot{d} = -\dot{x}\cos\theta_d - \dot{y}\sin\theta_d = -u_1\cos(\Delta\theta) \tag{5}$$

where $\Delta\theta$ is defined as $\Delta\theta \triangleq (\theta_d - \theta)$.

The task of converging the mobile robot from its present coordinates (x, y, θ) to the desired coordinates (x_d, y_d, θ_d) is equivalent to the task of converging the variables p, d and $\Delta\theta$ to zero. Indeed, if p, d and $\Delta\theta$ are simultaneously zero, it can be shown that $x = x_d$, $y = y_d$ and $\theta = \theta_d$.

In the next section we study the asymptotic convergence of the mobile robot to the line $\mathcal{L}$ and in the following section we design a controller for the asymptotic feedback stabilization of the robot to its desired posture.

3. Asymptotic Convergence to the Line

The mobile robot converges onto the line $\mathcal{L}$ if the variables p and $\Delta\theta$ converge to zero. To achieve this task, we use the following theorem by Mukherjee and Chen (1993).

Theorem 1: (Asymptotic Stability Theorem)
Consider the nonautonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}(t)) \tag{6}$$

where $\boldsymbol{f} : R_+ \times D \to R^n$ is a smooth vector field on $R_+ \times D$, $D \subset R^n$ is a neighborhood of the origin $\boldsymbol{x} = 0$. Let $\boldsymbol{x} = 0$ be an equilibrium point for the system described by Eq.(6). We then have

$$\boldsymbol{f}(t, 0) = 0, \qquad \forall t \geq 0 \tag{7}$$

(a) A necessary condition for stable systems
Let $V(t, \boldsymbol{x}) : R_+ \times D \to R_+$ be locally positive definite and analytic on $R_+ \times D$, such that

$$\dot{V}(t, \boldsymbol{x}) \triangleq \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \boldsymbol{x}}\right) \boldsymbol{f}(t, \boldsymbol{x}) \tag{8}$$

is locally negative semidefinite. Then whenever an odd derivative of V vanishes, the next derivative necessarily vanishes and the second next derivative is necessarily negative semidefinite.

(b) A sufficient condition for asymptotically stable systems
Let $V(\boldsymbol{x}) : D \to R_+$ be locally positive definite and analytic on D, such that $\dot{V} \leq 0$. If there exists a positive integer k such that

$$\begin{cases} V^{(2k+1)}(\boldsymbol{x}) < 0 & \forall \boldsymbol{x} \neq 0 : \dot{V}(\boldsymbol{x}) = 0 \\ V^{(i)}(\boldsymbol{x}) = 0 & \text{for } i = 2, 3, \cdots, 2k \end{cases} \tag{9}$$

where $V^{(*)}(\boldsymbol{x})$ denotes the $(*)-th$ time derivative of V with respect to time, then the system is asymptotically stable. However, if $V^{(j)}(\boldsymbol{x}) = 0$, $\forall j = 1, 2, \cdots, \infty$, then the sufficient condition for the autonomous system to be asymptotically stable is that the set

$$S = \{\boldsymbol{x} : V^{(j)}(\boldsymbol{x}) = 0,\ \forall j = 1, 2, \cdots, \infty\}$$

contains only the equilibrium point $\boldsymbol{x} = 0$.

Proof:
Please refer to the proof in (Mukherjee and Chen, 1993).

Remark 1: In simple words, this theorem states that when the first derivative of the Lyapunov function vanishes, the second derivative also vanishes and asymptotic stability can be concluded if the third derivative is negative-definite. If the third derivative is negative-semidefinite and also vanishes, the fourth derivative necessarily vanishes and asymptotic stability can be concluded from the negative-definiteness of the fifth derivative. This logic can be used recursively for higher order derivatives of the Lyapunov function.

Theorem 2: (Asymptotic Convergence to the line $\mathcal{L}$)
The control input u_2 defined by the relation

$$u_2 = p\,\mathrm{sinc}(\Delta\theta)\, u_1 + \alpha\Delta\theta, \qquad \alpha > 0 \tag{10}$$

where $\mathrm{sinc}(\Delta\theta) \triangleq \begin{cases} \sin(\Delta\theta)/\Delta\theta & \text{if } \Delta\theta \neq 0 \\ 1 & \text{if } \Delta\theta = 0 \end{cases}$
will asymptotically stabilize the mobile robot onto the line $\mathcal{L}$ for any bounded u_1 satisfying

$$\lim_{t\to\infty} \int_\tau^t \int_\tau^t \int_\tau^t u_1^2 = \infty \tag{11}$$

for all finite $\tau \geq 0$. Specifically, $(p, \Delta\theta) = (0, 0)$ is a globally uniformly asymptotically stable equilibrium point for the system

$$\begin{aligned} \dot{p} &= \sin(\Delta\theta)\, u_1 \\ \Delta\dot{\theta} &= -u_2 \end{aligned}$$

if u_1 and u_2 are chosen according to Eqs.(10) and (11).

Proof:
To prove the convergence of p and $\Delta\theta$ to zero, a continuously differentially Lyapunov function candidate V is defined as follows

$$V = \frac{1}{2}\left(p^2 + \Delta\theta^2\right) \tag{12}$$

Clearly, V is globally positive definite, radially unbounded and decrescent. The derivative of V can be computed as

$$\begin{aligned}\dot{V} &= p\,\dot{p} - \Delta\theta\,\dot{\theta} \\ &= -\Delta\theta\left[u_2 - p\,\frac{\sin(\Delta\theta)}{\Delta\theta}u_1\right] \\ &= -\alpha\,\Delta\theta^2 \;\leq 0 \end{aligned}\tag{13}$$

where Eqs.(1), (4) and (10) were substituted. Since $\dot{V}$ in Eq.(13) is globally negative semi-definite, the equilibrium point $(p, \Delta\theta) \equiv (0,0)$ is globally uniformly stable.

To prove asymptotic stability, we realize that the first derivative of V is negative semi-definite and vanishes on the set S, $S \triangleq \{(p, \Delta\theta) : \Delta\theta = 0\}$. The third derivative of V can be computed as $V^{(3)} = -2\alpha\left(\dot{\Delta\theta}\right)^2 = -2\alpha\, u_2{}^2$ and $V^{(3)} = -2\alpha\, p^2\, u_1{}^2$ on S. The term $-2\alpha\, p^2$ appearing in $V^{(3)}$ is negative definite on S. Following the same line of proof of Theorem 1 in (Mukherjee and Chen, 1993) we can conclude that p and $\Delta\theta$ converge to zero if the condition in Eq.(11) is satisfied.

Remark 2: The condition in Eq.(11) can be satisfied quite easily by almost any function $u_1(t)$. Some simple examples include exponentially convergent, constant, or periodic input u_1.

Remark 3: Though V is an explicit function of p and $\Delta\theta$, it is an implicit function of time. From the above theorem we can conclude that $V(t) \longrightarrow 0$ as $t \longrightarrow \infty$ if u_1 and u_2 satisfy Eqs.(10) and (11).

4. Stabilization to a Desired Posture

4.1 The Stabilizing Controller

The stabilizing controller consists of two control laws, u_1 and u_2. The control law for u_2 is proposed in Eq.(10) in terms of u_1. To define the control law for u_1, we let $t = 0$ be the initial point of time and σ be some positive number less than $\pi^2/8$. If $V(0) < \sigma$, we choose $T = 0$. Otherwise, we choose T such that $V(T) < \sigma$. The control law for u_1 is now proposed as follows

$$\text{if } \; t < T: \qquad u_1 = C \tag{14}$$

$$\text{if } \; t \geq T:$$

$$\begin{cases} u_1 = \left[\mu\, d\,\left(d^2 + \eta^2 - n\,V\right) + \omega\,\eta\right]/\cos(\Delta\theta) \\ \dot{\eta} \;= -\mu\,\eta\,\left(d^2 + \eta^2 - n\,V\right) + \omega\, d,\; \eta(T) \neq 0 \end{cases}\tag{15}$$

where, μ and n are strictly positive numbers, and ω and C are nonzero numbers. In the controller proposed above, α, σ, C, μ, n, ω and T are controller parameters.

Remark 4: In the general case, the controller given by Eq.(10) and Eq.(14) or (15) is piecewise smooth. However, there will be at most one switching between the control laws given by Eqs.(14) and (15). Specifically, there will be one switching if the initial value of V is greater than σ, otherwise there will be no switching and the controller will be smooth.

Remark 5: The term $\cos(\Delta\theta)$ in the controller in Eq.(15) will always be positive. This will be established later in this section. Therefore the controller in Eq.(15) is well defined. Note however, if $T > 0$ the control action u_1 may be discontinuous at time $t = T$ if the controller parameters are not chosen properly.

Remark 6: The control law for u_1 given by Eq.(14) was chosen quite arbitrarily. Any choice of u_1 that satisfies Eq.(11) could be chosen. A constant u_1 was chosen in Eq.(14) for the sake of simplicity.

Remark 7: The dynamic controller in Eq.(15) along with Eq.(5) represents a time-varying oscillator in the phase space of d, η. This oscillator is physically motivated and plays a key role in preventing the mobile robot from getting stuck at any point other than the desired posture. While a rigorous proof is provided in the next section, physical insight into the dynamics of the robot can be attained if we notice that the equilibrium point $(d, \eta) = (0,0)$ of the oscillator is unstable for all $V > 0$ and stable for $V = 0$. This implies that d, η converges to zero only after the mobile robot has converged onto the line $\mathcal{L}$. In the next section we show that in fact d, η and V converge to zero simultaneously and as a result the mobile robot is stabilized to its desired posture.

4.2 Stability Analysis

Before we prove the asymptotic stabilty of the of equilibrium point $(p, d, \Delta\theta) = (0,0,0)$, we present some new results with the help of the following theorem.

Theorem 3: (Time-Varying Oscillator)
Consider the following nonlinear time-varying oscillator

$$\begin{aligned} \dot{z}_1 &= -\mu\, z_1\, \left[z_1^2 + z_2^2 - n\,W(t)\right] - \omega\, z_2 \\ \dot{z}_2 &= -\mu\, z_2\, \left[z_1^2 + z_2^2 - n\,W(t)\right] + \omega\, z_1 \end{aligned} \qquad t \geq T \geq 0, \tag{16}$$

where $\mu > 0$, $n > 0$, and $\omega \neq 0$ are constants and $W(t)$ is a continuously differentiable function that satisfies $0 < W(t) < M$ $\forall t \geq 0$ for some constant $M > 0$. Let $\boldsymbol{z}(t) = [\, z_1(t) \quad z_2(t)\,]^T$ be any solution of the oscillator starting from an initial condition $\boldsymbol{z}(T) = (\, z_{10} \quad z_{20}\,)^T$. Then, the following statements are true

(a) $\boldsymbol{z}(t)$ is globally uniformly ultimately bounded,

(b) For any nonzero initial condition $\boldsymbol{z}(T) = (\, z_{10} \quad z_{20}\,)^T$, $\boldsymbol{z}(t)$ satisfies

$$\lim_{t\to\infty} \int_\tau^t\!\int_\tau^t\!\int_\tau^t \dot{z}_i^2 = \infty \qquad \forall\tau \geq T, \qquad i = 1, 2,$$

(c) If $W(t)$ is monotonically decreasing and $W(t) \longrightarrow 0$ as $t \longrightarrow \infty$, then $\boldsymbol{z}(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

Proof:

(a) Consider the scalar function

$$\tilde{V}(z, \dot{z}) = \frac{1}{2}\,\|\boldsymbol{z}^2\| \triangleq \frac{1}{2}\left(z_1^2 + z_2^2\right) \tag{17}$$

The function $\tilde{V}$ satisfies

1. $\tilde{V} \geq 0$,
2. $\tilde{V} \longrightarrow \infty$ uniformly for $t > 0$ as $\|\boldsymbol{z}\| \longrightarrow \infty$, and
3. $\tilde{V}$ has continuous first partials for z_1 and z_2.

The derivative of $\tilde{V}$ along the system trajectory given by Eq.(16) is computed as

$$\begin{aligned}\dot{\tilde{V}} &= z_1\,\dot{z}_1 + z_2\,\dot{z}_2 \\ &= -\mu\,\left(z_1^2 + z_2^2\right)\left[z_1^2 + z_2^2 - n\,W(t)\right] \\ &\leq -\mu\,\left(z_1^2 + z_2^2\right)\left[z_1^2 + z_2^2 - n\,M\right] \end{aligned}\tag{18}$$

Define the closed set $S = \left\{\boldsymbol{z} \in R^2 \;:\; \|\boldsymbol{z}\| \leq \sqrt{n\,M}\right\}$ and let $S_\epsilon \supset S$ denote the set of all points whose distance from S is less than ϵ for a small number ϵ. Let S^C denote the complement of S and S_ϵ^C denote the complement of S_ϵ. Now it can be easily seen from Eqs.(17) and (18) that $\tilde{V}$ and its derivative satisfy:

1. $\tilde{V}(\boldsymbol{z}_1) < \tilde{V}(\boldsymbol{z}_2)$ for all $\boldsymbol{z}_1 \in S$ and all $\boldsymbol{z}_2 \in S_\epsilon^C$, and
2. $\dot{\tilde{V}}(\boldsymbol{z}, t) \leq -\mu\,\epsilon^2\,(\epsilon^2 + n\,M) < 0$ for all $t \geq 0$ and all $\boldsymbol{z} \in S_\epsilon^C$

The global uniform ultimate boundedness of the system trajectories can now be concluded from (LaSalle and Lefschetz, 1961).

(b) For any given $\tau \geq T$, define

$$\begin{aligned} m &= \min_{T \leq t \leq \tau} W(t) > 0 \\ \delta \; &= \min\left\{\|\boldsymbol{z}(T)\|, \sqrt{nm}\right\} > 0 \\ S_\delta &= \left\{\boldsymbol{z} \in R^2 : \|\boldsymbol{z}\| \geq \delta\right\} \end{aligned}$$

Then on the boundary of S_δ, we have $\|z\| = \delta$, and from Eq.(18) we get

$$\begin{aligned}\dot{\tilde{V}} &= -\mu \|z\|^2 \left[\|z\|^2 - n\,W(t)\right] \\ &= -\mu \delta^2 \left[\delta^2 - n\,W(t)\right] \geq 0, \quad \forall t \in [T, \tau]\end{aligned}$$

since $n\,W(t) \geq \delta^2 \; \forall t \in [T, \tau]$. Therefore, all solutions $z(t)$ on the boundary of S_δ remains on the boundary of S_δ or moves inside S_δ over $t \in [T, \tau]$. Consequently, any trajectory starting inside S_δ, does not leave S_δ before $t = \tau$. Hence $\|z(\tau)\| = \delta$ and $\dot{z}_1, \dot{z}_2 \not\equiv 0$ for all $t \geq \tau$ because of the oscillator dynamics. Due to continuity of $\dot{z}_1, \dot{z}_2$, we have

$$\int_\tau^t \dot{z}_i^2 \, dt \geq \delta_1, \quad i = 1, 2$$

for some $\delta_1 > 0$, $\forall t$ sufficiently large. Therefore

$$\lim_{t \to \infty} \int_\tau^t \int_\tau^t \int_\tau^t \dot{z}_i^2 = \infty \qquad i = 1, 2,$$

(c) Consider a time series $t_0, t_1, t_2, t_3, \cdots$, where $T = t_0 < t_1 < t_2 < t_3 < \cdots$. Let the values of $W(t)$ at these instants of time be denoted as $M_0, M_1, M_2, M_3, \cdots$. In other words, $M_i = W(t = t_i)$, $i = 0, 1, 2, 3, \cdots$. Since $W(t)$ is nonincreasing, we know that $W(t) \leq M_i \; \forall \, t \in [t_i, \infty)$ and $M_i \geq M_{i+1}$, $i = 0, 1, 2, 3, \cdots$. Define the closed set $S_i = \{z \in R^2 \; : \; \|z\| \leq \sqrt{n M_i}\}$ and let $S_i \subset U_i = \{z \in R^2 \; : \; \|z\| \leq 2\sqrt{n M_i}\}$, $i = 0, 1, 2, 3, \cdots$. It simply follows that $U_{i+1} \subset U_i$, $i = 0, 1, 2, 3, \cdots$. We construct the time series recursively starting with $t_0 = T$. Assume t_i has been selected. Since $W_i \leq M_i \; \forall t \geq t_i$, the trajectories of the system given by Eq.(16) will be confined within U_i after a finite time interval starting at $t = t_i$ due to global uniform ultimate boundedness. Denote this time interval by t_i' and define

$$\begin{aligned} t_i'' &= \min\{t : W(t) \leq M_1/2\} \\ t_{i+1} &= t_i + \max\{t_i', t_i''\} \end{aligned}$$

With this time series, we have $M_{i+1} \leq M_i/2 \leq M_0/2^{i+1}$. Now we prove the convergence of $z(t)$ to zero. Given any $\epsilon > 0$, we can select I such that $\sqrt{n M_I} < \epsilon/2$. Then the trajectory of the system will be confined in $U_I \; \forall t > t_I$, *i.e.* $z(t) \in U_I \; \forall t > t_I$. Therefore $\|z(t)\| < \epsilon \; \forall t > t_I$ which implies $z(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

Remark 8: If the condition $W(t) \longrightarrow 0$ in (c) is not satisfied, then the trajectories of the system will eventually be confined within a closed set U always containing the equilibrium point. The size of this set depends on the limiting value of $W(t)$. If this is a small number, the closed set U will be a small region about the equilibrium. Then uniform ultimate boundedness is a practical notion of stability which is sometimes referred to as "practical stability" in the literature (LaSalle and Lefschetz, 1961; Spong and Vidyasagar, 1989).

We are now ready to prove the asymptotic convergence of the mobile robot to the desired posture.

Theorem 4: (Asymptotic Convergence to the Desired Posture)

The mobile robot system described by Eq.(1) globally asymptotically converges to its desired posture x_d, y_d, θ_d, *i.e.*, $(p, d, \Delta\theta) \longrightarrow (0, 0, 0)$ as $t \longrightarrow \infty$ with the controller described by Eq.(14) or (15) and Eq.(10).

Proof:

We first prove the theorem for the case (a) $T = 0$. Then we prove it for the case (b) $T > 0$.

Case (a): Since $T = 0$, $V(0) < \pi^2/8$. Therefore from Theorem 2, we know that $V(t) < \pi^2/8 \;\; \forall t > 0$ regardless of the choice of u_1. This implies that $|\Delta\theta| < \pi/2 \;\; \forall t > 0$. Hence the controller in Eq.(15) is well defined.

Now suppose that $V(0) \neq 0$. The closed loop sub-system defined by Eqs.(15) and (5) can be converted to the form given by Eq.(16) by letting $z_1 \triangleq d$, $z_2 \triangleq \eta$ and $W(t) = V(t)$. Using Theorem 3, part (a), we conclude that d and η are bounded. From part (b) we additionally know that

$$\lim_{t \to \infty} \int_\tau^t \int_\tau^t \int_\tau^t d^2 = \infty \quad \forall \tau \geq 0$$

From Eq.(5) and from the fact that $\cos \Delta\theta < 1$, we can then conclude

$$\lim_{t \to \infty} \int_\tau^t \int_\tau^t \int_\tau^t u_1^2 = \infty \quad \forall \tau \geq 0$$

Thus from Theorem 2 we can conclude that $W(t) = V(t) \longrightarrow 0$ as $t \longrightarrow \infty$. This in turn implies $p(t)$, $\Delta\theta(t)$, $d(t)$, $\eta(t) \longrightarrow 0$ as $t \longrightarrow \infty$ from the definition of V in Eq.(12) and part (c) of Theorem 3.

If $V(0) = 0$, then $(p, \Delta\theta) = (0, 0) \; \forall t \geq 0$, from Theorem 2. The closed loop subsystem defined by Eqs.(15) and (5) is now of the form

$$\begin{aligned} \dot{d} &= -\mu\, d\, (d^2 + \eta^2) - \omega\, \eta \\ \dot{\eta} &= -\mu\, \eta\, (d^2 + \eta^2) + \omega\, d \end{aligned}$$

Clearly, $d(t)$, $\eta(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

Case (b): To prove the theorem for case (b), we only need to show that under the choice of u_1 given by Eq.(14), (i) there is no finite escape for d and (ii) $V(t)$ will be become less than $\pi^2/8$ in finite time. By substituting Eq.(14) into Eq.(5) the behavior of d is seen to be governed by

$$\dot{d} = -C \cos \Delta\theta$$

Clearly, d has no finite escape time. Furthermore, it can be easily verified that u_1 in Eq.(14) satisfies the condition in Eq.(11). Therefore $V(t)$ will become less than $\pi^2/8$ in finite time since it tends to zero as $t \longrightarrow \infty$.

5. Simulation Results

In this section simulation results are presented that amply demonstrate the capability of the controller to stabilize the mobile robot to different desired postures. For these cases, the initial and final coordinates of the robot are given in Table 1, where the units are meters and degrees. The choice of the controller parameters and the total simulation time, t_f, for these four cases are shown in Table 2 in S.I. units.

Case A represents the parallel parking problem. The trajectory of the robot in the x-y plane for this case is shown in Fig.2. It is clear from this figure that the robot converges to its desired posture. The value of the Lyapunov function at the initial point of time was less than $\sigma = 0.8$ and hence the controller was smooth. The total simulation time required was 4.19 seconds. In Cases B, C and D, the robot is required to make a 360 degree, a 135 degree and a -180 degree turn respectively, between its initial and final configurations. In each of these three cases the initial value of the Lyapunov function was larger than the value of $\sigma = 0.8$. Therefore the controllers in each of these cases had one switching. The switching time $t = T$ for the controllers are given in Table 2. The trajectory of the robot for cases B, C and D are shown in Figs.3, 4 and 5 respectively. These figures clearly demonstrate that the mobile robot converges to its desired posture in each case. The total simulation time required for cases B, C and D were 8.68, 9.00 and 6.52 seconds respectively.

6. Conclusion

In this paper a piecewise smooth dynamic controller is proposed for the point-to-point stabilization of a two-wheeled nonholonomic mobile robot. The piecewise smooth nature of the controller is derived from the fact that the controller switches once after a certain measure of the error in the posture has reduced below some constant value. For problems where this error is initially low, there is no switching and the control action generated is smooth. The controller proposed herein is obtained as a solution to a first order differential equation and as such it can be classified as a dynamic controller. In the literature few results can be found on feedback stabilization of nonholonomic systems using a dynamic controller. In our case, the dynamic controller is simple in its formulation and utilizes the dynamics of a nonlinear oscillator that plays a key role in preventing the mobile robot from getting stuck at any point other than the desired posture. The proposed controller is physically motivated and guarantees the asymptotic stabilization of the states of the mobile robot to their desired values. Simulation results confirm the efficacy of the stabilizing control.

Acknowledgement

The first and second authors gratefully acknowledge the support provided by the National Science Foundation, NSF Grant No. CMS-9410157 and ECS-9410646, respectively, in carrying out this research.

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Table 1: Initial and Final Configurations of the Mobile Robot

	$(x_i \quad y_i \quad \theta_i)$	$(x_d \quad y_d \quad \theta_d)$
Case A	(0 0 0)	(0 1 0)
Case B	(0 0 0)	(1 1 360)
Case C	(0 0 0)	(−5 3 135)
Case D	(0 0 0)	(1 0 −180)

Table 2: Controller Parameters and Simulation Time

	α	ω	μ	n	σ	C	T	t_f
Case A	3.00	1.50	0.985	50.0	0.80	1.0	0.00	4.19
Case B	1.00	1.00	0.350	9.00	0.80	1.5	2.72	8.68
Case C	3.35	1.25	0.750	50.0	0.80	−3.0	0.31	9.00
Case D	3.00	1.00	0.500	25.0	0.80	3.0	0.33	6.52

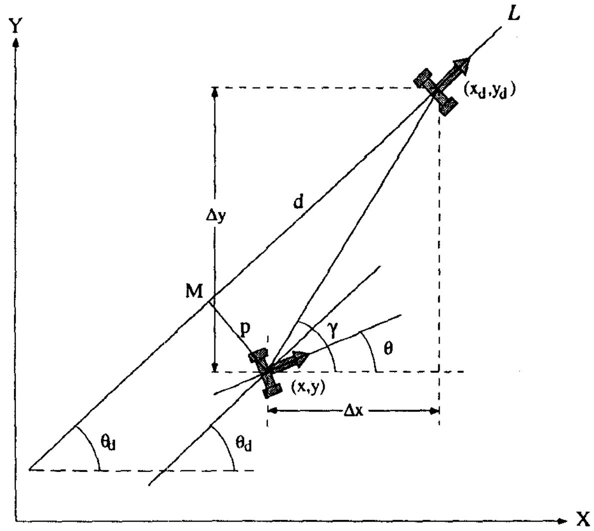

Fig.1 Diagram showing the initial and final configration of the mobile robot

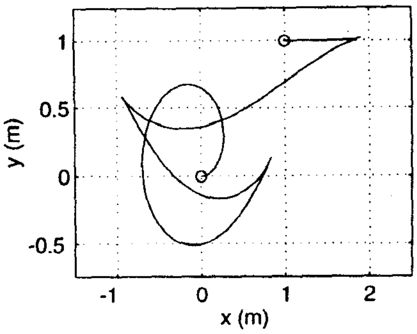

Fig.3 Trajectory of the mobile robot for Case B

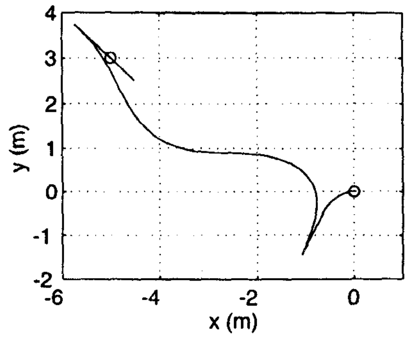

Fig.4 Trajectory of the mobile robot for Case C

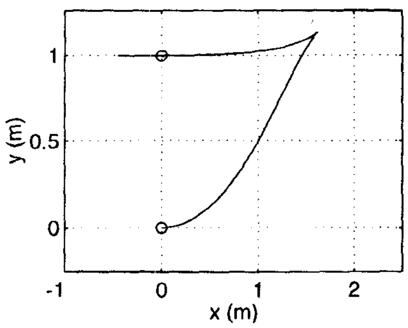

Fig.2 Trajectory of the mobile robot for Case A

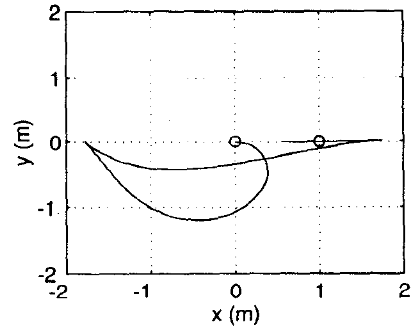

Fig.5 Trajectory of the mobile robot for Case D