Dilation-5 Embedding of 3-Dimensional Grids into Hypercubes

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Abstract

We present an algorithm to map the nodes of a 3-dimensional grid to the nodes of its optimal hypercube on a one-to-one basis with dilation at most 5.

1 Introduction

A ***binary hypercube of dimension n***, is an undirected graph of 2^n nodes labeled 0 to $2^n - 1$ in binary where two nodes are connected if and only if their labels differ in exactly one bit position. Since a hypercube has a regular structure with a rich interconnection, it is a popular multiprocessor computer architecture. An embedding for a 3-D grid into a hypercube can be viewed as a high level description of an efficient method to simulate an algorithm designed for a parallel computer with a 3-D grid structure on a parallel computer with a hypercube structure. Here we are interested in the problem of mapping the nodes of any 3-D grid into the nodes of its optimal hypercube (the smallest hypercube with at least as many nodes as the grid) on a one-to-one basis, so that dilation (the worst case distance between grid-neighbours in the hypercube) is bounded by a small constant.

It is known that every 2-D grid can be embedded into its optimal hypercube with at most dilation 2 [C1, C2], [HLV]. This result is optimal as it has been proven that over 38 percent of all 2-D grids need at least dilation 2 [BS]. However, not much is known about the optimal dilation for embedding 3-D grids into optimal hypercubes. A non-trivial extension of the technique in [C1, C2] for embedding 3-D grids into optimal hypercubes with at most dilation 7 was given by Chan in [C3]. A dilation-6 embedding scheme was derived later in [LH]. In this paper, we introduce a simple dilation-5 embedding strategy for embedding 3-D grids into optimal hypercubes.

2 General Outline

Consider a 3-D grid G of size $\alpha \times \beta \times \gamma$. Our objective is to label each node of G with a unique $\lceil \log_2 \alpha\beta\gamma \rceil$-bit binary number, which effectively names the node in the optimal $\lceil \log_2 \alpha\beta\gamma \rceil$-cube to which it is mapped. G can be seen as comprising of γ layers of 2-D grids each of size $\alpha \times \beta$. Let $l = 2^{\lfloor \log_2 \alpha\beta \rfloor}$. To aid the assignment of binary labels, we will partition G's nodes into l groups, called ***links***, evenly in the sense that when counting from layer 0 to layer k ($0 \leq k \leq \gamma - 1$), the number of nodes belonging to any particular link is either $\lfloor (k+1)\alpha\beta/l \rfloor$ or $\lceil (k+1)\alpha\beta/l \rceil$.

Partitioning G's nodes into l links is equivalent to determining a unique pair of numbers for each node of G, namely, a ***link-number*** and a ***bead-number***. A node's link-number indicates the link to which a node belongs, while its bead-number tells its position in that link. After the partitioning, we will use a node's link-number to determine the first $\log_2 l$ bits of its binary label, which we will call the ***link-label***. And we will use a node's bead-number to determine the remaining bits of its binary label, which we will call the ***bead-label***.

3 Preliminaries

In [HLV], a general embedding strategy for embedding 2-D grids into 2-D grids (of different sizes) was introduced. In particular, it can be used to embed an $\alpha \times \beta$ guest grid into an $\alpha' \times \beta'$ host grid where $\alpha' = 2^{\lfloor \log_2 \alpha \rfloor}$ and $\beta' = \lceil \alpha\beta/\alpha' \rceil$. (Note that in [HLV], the embedding strategy was described for embedding a $h \times w$ guest grid into a $h' \times w'$ host grid where $w' \leq w$ and $h' = \lceil hw/w' \rceil$, here we swapped the roles of row and column for later convenience.)

As the embedding is one-one, $\alpha'\beta'$ has to be greater than $\alpha\beta$, and some nodes in the $\alpha' \times \beta'$ host grid do not correspond to any node in the $\alpha \times \beta$ guest grid. A nice property of this embedding is that any unmapped node is only found in the last column of the $\alpha' \times \beta'$

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host grid. So we can view the above embedding as one that embeds an $\alpha \times \beta$ grid into a jagged grid of α' rows where each row consists of either β' or $\beta'-1$ nodes, and the total number of nodes of the jagged grid is exactly $\alpha\beta$. Figure 1 shows an example of such an embedding. It is proven in [HLV] that this method yields a dilation-2 embedding for any $\alpha \times \beta$ guest grid. A careful study of the proof will reveal that the method actually ensures that any two neighbouring nodes in the $\alpha \times \beta$ grid can only be mapped to one the following 5 sets of relative positions in the α'-row jagged grid: {[x, y], [x, y+1]}, {[x, y], [x, y+2]}, {[x, y], [x+1, y]}, {[x, y], [x+1, y-1]}, {[x, y], [x+1, y+1]} where $[x, y]$ denotes the position in row x, column y of the jagged grid. We will utilize this 2-D grid embedding method as the first step of our embedding stratedgy, and we will refer to it as the *trio's method.*

The process of partitioning G into l links depends on a length-l vector of 1's and 2's, $v(\alpha,\beta)$ [C2].

Definition 1 *Define*

$$v(\alpha,\beta) = [v_0, v_1, v_2, \ldots, v_{l-1}] = \begin{bmatrix} \lceil \alpha\beta/l \rceil \\ \lfloor \alpha\beta/l \rfloor \\ \lfloor 2\alpha\beta/l \rfloor - \lfloor \alpha\beta/l \rfloor \\ \lfloor 3\alpha\beta/l \rfloor - \lfloor 2\alpha\beta/l \rfloor \\ \vdots \\ \lfloor (l-1)\alpha\beta/l \rfloor - \lfloor (l-2)\alpha\beta/l \rfloor \end{bmatrix}^T$$

where $l = 2^{\lfloor log_2 \alpha\beta \rfloor}$

Basically, v is defined so that the 2's are evenly distributed among the 1's when there are more 1's than 2's and vice versa when there are more 2's than 1's. The vector v has a Cyclic Sum Property which is stated below.

Definition 2 *For* $0 \le s \le l-1$ *and* $k \ge 0$, *define*

$$CYCLIC\text{-}SUM(s,k) = \sum_{i=s}^{s+k-1} v_{i \bmod l} [C2]$$

Fact 1 (Cyclic Sum Property)
$\lfloor k\alpha\beta/l \rfloor \le CYCLIC\text{-}SUM(s,k) \le \lceil k\alpha\beta/l \rceil$
for $0 \le s \le l-1$, $k \ge 0$

To determine the final binary label given to each node the binary reflected Gray code sequence is used.

Definition 3 *For* $t \ge 0$ *and* $0 \le p \le 2^t - 1$, *define*

$$GRAY(t,p) = (p+1)\textit{th element of the t-bit binary reflected Gray code sequence}$$

For example, $GRAY(3,5) \equiv 111$ since 111 is the 6th element of (000, 001, 011, 010, 110, 111, 101, 100).

Fact 2 (Gray Code Property) *In the t-bit binary reflected Gray code sequence, for any p such that* $0 \le p \le 2^t - 1$ *and for any* $i \ge 0$, *the number of differing bits of* $GRAY(t,p)$ *and* $GRAY(t, (p \pm i) \bmod 2^t)$ *is at most* i.

4 Dilation-5 Embedding Strategy

The following steps are illustrated by Figures 1 to 4 using a $5 \times 5 \times 5$ grid as an example.
Let $l = 2^{\lfloor \log_2 \alpha\beta \rfloor}$, $\tilde{\alpha} = 2^{\lfloor \log_2 \alpha \rfloor}$ and $\tilde{\beta} = l/\tilde{\alpha}$.

1. Transform to γ layers of jagged grids:
   Using the trio's algorithm, transform all γ layers of $\alpha \times \beta$ 2-D grids into γ layers of identical 2-D jagged grids of $\tilde{\alpha}$ rows. (See Figure 1)
2. Partition each jagged layer into cells:
   Imagine there is a super-chain spanning all the nodes of a layer for each of the γ jagged layers. Divide each super-chain, hence jagged layer, into l cells according to vector $v(\alpha,\beta)$ and label the cells from 0 to $l-1$. Therefore the number of nodes in cell i should be equal to v_i $(0 \le i \le l-1)$. (See Figure 2)
3. Determine the link-number of each node:
   For any node $\mathcal{N}$, if $\mathcal{N}$ is in cell c $(0 \le c \le l-1)$ of layer k $(0 \le k \le \gamma - 1)$, its link-number is
   $$LINK(\mathcal{N}) = (c-k) \bmod l$$
   (See Figure 3)
4. Determine the bead-number of each node:
   For any node $\mathcal{N}$, if $\mathcal{N}$ is in cell c $(0 \le c \le l-1)$ of layer k $(0 \le k \le \gamma - 1)$, its bead-number is
   $$BEAD(\mathcal{N}) = CYCLIC\text{-}SUM(LINK(\mathcal{N}), k) + \delta(\mathcal{N})$$
   where δ is defined as follows: let $t = \left\lfloor c/(\tilde{\beta}+2) \right\rfloor$ if t is even (odd),
   $$\delta(\mathcal{N}) = \begin{cases} 1 & \text{if } \mathcal{N} \text{ has an immediately preceding (succeeding) node } \mathcal{M} \text{ in its super-chain such that } LINK(\mathcal{M}) = LINK(\mathcal{N}) \\ 0 & \text{otherwise} \end{cases}$$
   (See Figures 3 and 4)

5. Determine the link-label of a node:
   For any node $\mathcal{N}$, define
   $LK1(\mathcal{N}) = \left\lfloor LINK(\mathcal{N})/\tilde{\beta} \right\rfloor$ and $LK2(\mathcal{N}) = LINK(\mathcal{N}) \bmod \tilde{\beta}$, the link-label of $\mathcal{N}$ is

   $$GRAY(\log_2 \tilde{\alpha}, LK1(\mathcal{N}))GRAY(\log_2 \tilde{\beta}, LK2(\mathcal{N}))$$

6. Determine the bead-label of a node:
   For any node $\mathcal{N}$, its bead-label is

   $$GRAY(\lceil \log_2 \alpha\beta\gamma \rceil - \log_2 l, BEAD(\mathcal{N}))$$

7. Concatenate the link-label and bead-label to get the complete binary label for every node.

5 Dilation Analysis

We will call any neighbouring nodes in the same layer of a 3-D grid G *horizontal neighbours* and any neighbouring nodes at the same position of 2 adjacent layers of G *vertical neighbours.*

Let us consider horizontal neighbours first.

For any horizontal neighbours $\mathcal{N}_1$ and $\mathcal{N}_2$ of the grid G, they must be mapped to the same jagged grid. Moreover, the trio's method will map them to one of the following 5 sets of relative positions: {[x, y], [x, y+1]}, {[x, y], [x, y+2]}, {[x, y], [x+1, y-1]}, {[x, y], [x+1, y]}, {[x, y], [x+1, y+1]}.

WLOG, for Lemma 1 to Lemma 5 assume that $\mathcal{N}_1$ is mapped to position $[x, y]$ of a jagged layer, and $\mathcal{N}_1$ is in cell p, $\mathcal{N}_2$ is in cell q ($0 \leq p, q \leq l-1$).

Lemma 1 *If $\mathcal{N}_2$ is mapped to $[x, y+1]$ or $[x, y+2]$, then $p \leq q \leq p+2$.*

Proof outline: Note that each cell contains 1 or 2 nodes. If $\mathcal{N}_2$ is mapped to $[x, y+1]$, then $p \leq q \leq p+1$. If $\mathcal{N}_2$ mapped to $[x, y+2]$, then $p+1 \leq q \leq p+2$. □

Lemma 2 *If $\mathcal{N}_2$ is mapped to $[x+1, y-1]$, $[x+1, y]$ or $[x+1, y+1]$, then $p+\tilde{\beta}-2 \leq q \leq p+\tilde{\beta}+2$.*

Proof outline: We can prove this lemma by counting the number of nodes from $\mathcal{N}_1$ to $\mathcal{N}_2$ in their superchain and using the Cyclic Sum Property of the vector v used in constructing the cells. □

Lemma 3 *The number of differing bits of the link-labels of $\mathcal{N}_1$ and $\mathcal{N}_2$ is at most 3 if $p \leq q \leq p+2$ or $p+\tilde{\beta}-2 \leq q \leq p+\tilde{\beta}+1$, and is at most 4 if $q = p+\tilde{\beta}+2$.*

Proof outline: Let $q = p + i$ ($i = 0, 1, 2, \tilde{\beta}-2, \tilde{\beta}-1, \tilde{\beta}, \tilde{\beta}+1$ or $\tilde{\beta}+2$). It can be proved that

$$\begin{aligned} LK1(\mathcal{N}_2) &= (LK1(\mathcal{N}_1) + r) \bmod \tilde{\alpha} \\ &\quad \text{where } r = \left\lfloor i/\tilde{\beta} \right\rfloor \text{ or } \left\lceil i/\tilde{\beta} \right\rceil \\ LK2(\mathcal{N}_2) &= (LK2(\mathcal{N}_1) + i \bmod \tilde{\beta}) \bmod \tilde{\beta} \\ \textit{and} &\quad (LK2(\mathcal{N}_1) + (\tilde{\beta} - i) \bmod \tilde{\beta}) \bmod \tilde{\beta} \end{aligned}$$

Substituting i and using the Gray Code Property, the results can be proved. □

Lemma 4 *The number of differing bits of the bead-labels of $\mathcal{N}_1$ and $\mathcal{N}_2$ is at most 2.*

Proof outline: By the definition of bead-number and the Cyclic Sum Property, $|BEAD(\mathcal{N}_1) - BEAD(\mathcal{N}_2)| \leq 2$. Hence the result by the Gray Code Property. □

Lemma 5 *If $q = p + \tilde{\beta} + 2$, the number of differing bits of the bead-labels of $\mathcal{N}_1$ and $\mathcal{N}_2$ is at most 1.*

Proof outline: Note that one of $\left\lfloor p/(\tilde{\beta}+2) \right\rfloor$ and $\left\lfloor q/(\tilde{\beta}+2) \right\rfloor$ must be odd and the other must be even. (This is the reason why the factor $\tilde{\beta}+2$ is used in defining the function δ.) Then it can be proved that $|BEAD(\mathcal{N}_1) - BEAD(\mathcal{N}_2)| \leq 1$ using the Cyclic Sum Property. Hence the result by the Gray Code Property. □

Now, let us consider vertical neighbours.

For the following 2 lemmas, let $\mathcal{M}$ and $\mathcal{N}$ be any vertical neighbours such that $\mathcal{M}$ is above $\mathcal{N}$. Since all layers are transformed into jagged grids in the same way, $\mathcal{M}$ and $\mathcal{N}$ will be mapped to the same position of 2 adjacent jagged grids.

Lemma 6 *The number of differing bits of the link-labels of $\mathcal{M}$ and $\mathcal{N}$ is at most 2.*

Proof outline: With some arithmetic manipulation, we can show that $LK1(\mathcal{M}) = LK1(\mathcal{N}) \bmod \tilde{\alpha}$ or $(LK1(\mathcal{N}) + 1) \bmod \tilde{\alpha}$ and $LK2(\mathcal{M}) = (LK2(\mathcal{N}) + 1) \bmod \tilde{\beta}$. Thus the number of differing bits in both components of the link-labels of $\mathcal{M}$ and $\mathcal{N}$ are at most 1. □

Lemma 7 *The number of differing bits of the bead-labels of $\mathcal{M}$ and $\mathcal{N}$ is at most 2.*

Proof outline: Since $\mathcal{M}$ and $\mathcal{N}$ are mapped to the same position of 2 jagged grids, $\delta(\mathcal{M}) = \delta(\mathcal{N})$. So

$BEAD(\mathcal{N}) - BEAD(\mathcal{M}) = 1$ or 2. Hence the result by the Gray Code Property. □

The number of differing bits of the binary labels is at most 5 for any horizontal neighbours in G by Lemmas 3 to 5, and is at most 4 for any vertical neightbours by Lemmas 6 and 7. So the strategy described gives a dilation-5 embedding for any 3-D grid G.

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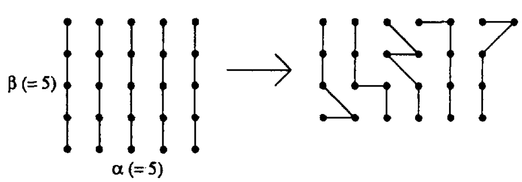

Figure 1: Transform a rectangular grid into a jagged grid by the trio's method

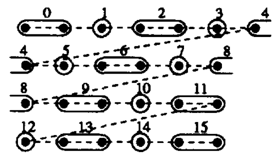

Figure 2: Partition of a jagged grid into l cells. Note that the dotted line represents the super-chain and each small circle or oval represents a cell. $v(5,5) = [2,1,2,1,2,1,2,1,2,2,1,2,1,2,1,2]$

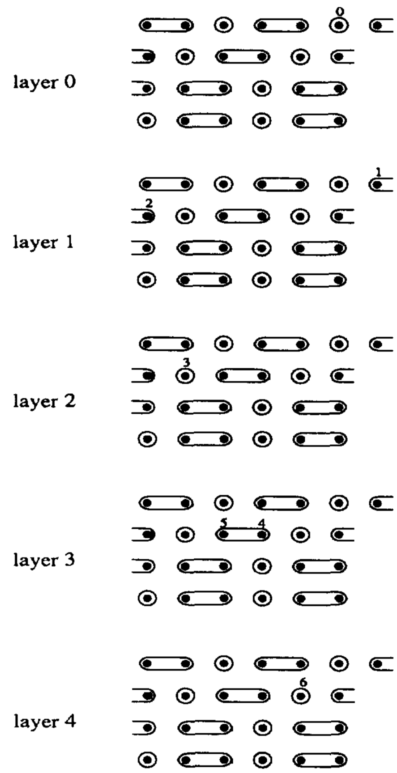

Figure 3: An example of link-number and bead-number assignments. Nodes of link 3 are shown with their bead-numbers

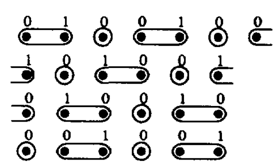

Figure 4: Values of δ function for the nodes of one layer