A Finite Energy Property of Stable Inversion to Nonminimum Phase Nonlinear Systems

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***Abstract*—Stable inversion is a completely new approach to the output tracking control of nonminimum phase nonlinear systems. It not only offers exact reproduction of a given reference output trajectory but also guarantees stability of all external and internal signals. After a brief review of the stable inversion problem, the paper establishes a finite energy property of stable inverses. That is, out of infinitely many possible inverse solutions, the one provided by the stable inversion process is the only one that has finite energy, or, the $\mathcal{L}_2(-\infty, +\infty)$-norm. The effectiveness of the stable inversion approach in output tracking as compared with the well-known classic inversion and nonlinear regulation approaches is demonstrated by working out an example of a nonlinear nonminimum phase system.**

***Index Terms*—Inversion, nonlinear, nonminimum phase, output tracking.**

I. Introduction

The stable inversion approach was first developed by Chen and Paden [3] to attack a very important and very difficult problem in nonlinear control: output tracking control of nonminimum phase systems. In this approach, the output tracking controller has a structure of feed-forward plus feedback. The nonminimum phase system is first stably inverted offline to obtain desired (and stable) state and input trajectories that satisfy the system dynamics equation and map exactly into a given desired output trajectory. Then the desired input is used as a feed-forward signal and the state error deviating from the desired state is used as a feedback signal to a stabilizing tracking controller. The consequence of this strategy is remarkably accurate output tracking together with guaranteed stability of both external and internal signals. This approach can be easily applied to many important engineering systems that are known to be nonminimum phase systems, such as airplanes, rockets, flexible robots, and more.

Stable inversion is closely related to two other approaches in output tracking control of nonlinear systems. The first is the classic inversion approach that controls the transient behavior precisely by using stabilizing feedback together with feed-forward signals generated by an inverse system. The classic inversion was first studied by Brockett and Mesarovic [1]. Later, Silverman developed an easy-to-follow step-by-step procedure for the inversion of a class of linear multivariable systems [8]. These linear results were extended to nonlinear real-analytic systems by Hirschorn [4] and Singh [9]. For a given desired output and a fixed initial condition, all these inversion algorithms produce causal inverses that are unbounded for nonminimum phase systems.

Also closely related is the nonlinear regulation approach recently developed by Isidori and Byrnes [5]. This approach also uses the structure of feed-forward plus feedback and it provides asymptotic output tracking for a class of reference trajectories generated by a given autonomous exosystem. The feed-forward signals are calculated by solving a set of nonlinear partial differential equations of the same order as the forward system dynamics. Besides the numerical tractability of nonlinear partial differential equations, a major concern is the possibly large transient error that is not controlled in this approach.

The stable inversion approach is designed to achieve the salient features of both classic inversion and nonlinear regulation and at the same time avoid the drawbacks of both. The price one has to pay is that the inversion process is noncausal or anticipatory, which is perfectly fine from an engineering point of view since the inverse system is not expressed as a solution to differential equations but as a general nonlinear mapping. For a class of nonlinear systems with a well-defined relative degree and hyperbolic zero dynamics, the problem of computing the stable inverses for nonminimum phase systems is converted into a corresponding two-point boundary value problem. In this paper, a very important feature, the finite energy property, of these stable inverses is investigated. This property suggests many numerical iterative methods to solve the two-point boundary value problems. The remainder of the paper is organized as follows. Section II briefly reviews the stable inversion problem and its corresponding two-point boundary value problem. The main contribution of the paper is Section III, establishing that out of infinitely many possible inverse solutions the stable inverse is the only one yielding a finite $\mathcal{L}_2(-\infty, +\infty)$-norm. An example is given in Section IV to demonstrate the effectiveness of the stable inversion approach in output tracking control as compared with the classic inversion and nonlinear regulation approaches. Finally, a conclusion is given in Section V.

II. Stable Inversion

Consider multivariable nonlinear control systems of the form

$$\dot{x} = f(x) + g(x)u \quad (1)$$

$$y = h(x) \quad (2)$$

where x is defined on an open neighborhood X of the origin of $\mathbb{R}^n$ and $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$. The mappings $f(x)$ and $g_i(x)$ [the ith column of $g(x)$] for $i = 1, 2, \cdots, m$ are smooth vector fields defined on X, and $h_i(x)$ [the ith component of $h(x)$] for $i = 1, 2, \cdots, m$ are smooth functions on X. Without loss of generality, it is assumed that $f(0) = 0$ and $h(0) = 0$. For such systems, the stable inversion problem as posed in [3] is as follows.

Stable Inversion Problem: Given a smooth reference output trajectory $y_d(t)$, find a control input $u_d(t)$ and a state trajectory $x_d(t)$ such that

1) $u_d(t)$ and $x_d(t)$ satisfy the differential equation

$$\dot{x}_d(t) = f[x_d(t)] + g[x_d(t)]u_d(t);$$

2) exact output tracking is achieved

$$h[x_d(t)] = y_d(t);$$

3) $u_d(t)$ and $x_d(t)$ are bounded and whenever $y_d(t) \to 0$ as $t \to \pm\infty$

$$u_d(t) \to 0, \quad x_d(t) \to 0 \quad \text{as} \quad t \to \pm\infty.$$

Here the pair $[u_d(t), x_d(t)]$ is referred to as the stable inverse solution for a given reference output $y_d(t)$. It is called stable inverse because of the boundedness and convergence provided by Condition 3). Besides, $x_d(t)$ is called the desired state trajectory and $u_d(t)$ the nominal control input.

For the time being, we only consider nonlinear systems satisfying the following assumption.

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Assumption 1:

1) The nonlinear system has a well-defined vector relative degree $r = [r_1, \cdots, r_m]^T \in \mathbb{N}^m$ at the origin.
2) The equilibrium point of the zero dynamics at the origin is hyperbolic.

To solve the stable inversion problem, the nonlinear system is first partially linearized. Define

$$\xi \stackrel{\text{def}}{=} [\xi_1^1, \xi_2^1, \cdots, \xi_{r_1}^1, \xi_1^2, \cdots, \xi_{r_2}^2, \cdots, \xi_{r_m}^m]^T$$
$$\stackrel{\text{def}}{=} [y_1, \dot{y_1}, \cdots, y_1^{(r_1-1)}, y_2, \cdots, y_2^{(r_2-1)}, \cdots, y_m^{(r_m-1)}]^T. \quad (3)$$

Choose η such that

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \Phi(x)$$

forms a change of coordinates with $\Phi(0) = 0$. In this new coordinate system, the system dynamics equation becomes

$$y^{(r)} = \alpha(\xi, \eta) + \beta(\xi, \eta)u \quad (4)$$
$$\dot{\eta} = q_1(\xi, \eta) + q_2(\xi, \eta)u \quad (5)$$

where

$$\alpha(\xi, \eta) = L_f^r h[\Phi^{-1}(\xi, \eta)]$$
$$\beta(\xi, \eta) = L_g L_f^{r-1} h[\Phi^{-1}(\xi, \eta)]$$

and q_1 and q_2 are defined accordingly. Set $y(t) \equiv y_d(t)$. Then, we immediately have

$$\xi = \xi_d$$
$$\stackrel{\text{def}}{=} [y_{d1}, \dot{y}_{d1}, \cdots, y_{d1}^{(r_1-1)}, y_{d2}, \cdots, y_{d2}^{(r_2-1)}, \cdots, y_{dm}^{(r_m-1)}]^T$$

and $y^{(r)}(t) = y_d^{(r)}(t)$. Solving for u from (4), we obtain

$$u \stackrel{\text{def}}{=} [\beta(\xi_d, \eta)]^{-1}[y_d^{(r)}(t) - \alpha(\xi_d, \eta)]. \quad (6)$$

Upon substituting this, (5) becomes the so-called *reference dynamics*

$$\dot{\eta} = p[y_d^{(r)}, \xi_d, \eta] \quad (7)$$

where

$$p[y_d^{(r)}, \xi_d, \eta] \stackrel{\text{def}}{=} q_1(\xi_d, \eta) + q_2(\xi_d, \eta)[\beta(\xi_d, \eta)]^{-1} \cdot [y_d^{(r)} - \alpha(\xi_d, \eta)].$$

The reference dynamics becomes autonomous *zero dynamics* when output $y_d(t)$ is set identically to zero. It has been shown that, for systems satisfying Assumption 1, the stable inverse pair, the desired state trajectory $x_d(t)$, and the nominal control input $u_d(t)$ can be constructed as follows [3]:

$$x_d = \Phi^{-1}(\xi_d, \eta_d) \quad (8)$$
$$u_d = [\beta(\xi_d, \eta_d)]^{-1}[y_d^{(r)} - \alpha(\xi_d, \eta_d)] \quad (9)$$

where η_d solves the following two-point boundary value problem:

$$\dot{\eta} = p[y_d^{(r)}, \xi_d, \eta] \quad (10)$$

subject to

$$\begin{cases} B^s[\eta(t_0)] = 0 \\ B^u[\eta(t_f)] = 0. \end{cases} \quad (11)$$

Here $B^s(\eta) = 0$ characterizes the local unstable manifold and $B^u(\eta) = 0$ the local stable one of the origin of the zero dynamics, and t_0 and t_f are the time which will be specified in the coming Assumption.

In order to guarantee the existence of solutions to this two-point boundary value problem, the reference output trajectory needs to satisfy mild sufficient conditions as follows.

Assumption 2:

1) $y_d(t) \in \mathbf{C}^r(-\infty, +\infty)$.
2) There exists $[t_0, t_f]$, a finite closed interval in $\mathbb{R}$, such that $y(t) \equiv 0$ for all $t \notin [t_0, t_f]$.
3) $\sup_{t\in[t_0, t_f]} \| [\xi_d^T(t), y_d^{(r)}(t)]^T \|_2$ is sufficiently small.

It has also been shown that the two-point boundary value problem locally has a unique solution when the nonlinear system and the reference trajectory satisfy Assumptions 1 and 2, respectively [7].

It is noticed that the reference trajectories are restricted to a compact interval. However, this is not a significant restriction in practice since all practical trajectories have a finite duration.

As mentioned earlier, the signals, $x_d(t)$ and $u_d(t)$, are incorporated in our output tracking controller to achieve exact output reproduction but at the same time maintain closed-loop stability. Even for a mismatched initial condition, asymptotic tracking is guaranteed as long as the forward system has been stabilized. This property is shared by both stable inversion and nonlinear regulation as both approaches have the same closed-loop structure. See [2] for more details on closed-loop controller design using stable inversion.

III. Finite Energy Property

The goal of this section is to establish that out of an infinite number of input and state trajectories that are capable of producing exactly a given desired output trajectory, the desired state trajectory and the nominal control input given by the stable inversion approach is the only pair yielding a finite $\mathcal{L}_2(-\infty, +\infty)$-norm. This is a very important property of stable inversion. It immediately suggests its value in many applications where output tracking, input energy consumption, internal vibration, etc., are of concern.

Before we start, let us recall two standard theorems from theory of ordinary differential equations. The first theorem concerns the local property of solutions on the stable or unstable manifolds of a hyperbolic equilibrium point.

Theorem 1: (See Wiggins [10] for a proof.) Let W^s and W^u be the local stable and unstable manifolds of a hyperbolic equilibrium point of a dynamic system. Then the solutions of the dynamic system with initial conditions in W^s (respectively W^u) approach the equilibrium point at an exponential rate asymptotically as $t \rightarrow +\infty$ (respectively, $t \rightarrow -\infty$).

The second theorem addresses the local property of solutions that are on neither the stable nor the unstable manifolds of a hyperbolic equilibrium point. To state the theorem, let the origin be the hyperbolic equilibrium point of a dynamic system. Denote by $B(h)$ a spherical neighborhood with center at the origin and radius of h.

Theorem 2: (See Miller and Michel [6] for a proof.) Let W^s and W^u be the local stable and unstable manifolds of a hyperbolic equilibrium point of a dynamic system. Then there exists a $\delta_1 > 0$ (respectively $\delta_2 > 0$) such that if $[\tau, \eta(\tau)] \in \mathbb{R} \times B(\delta_1)$ [respectively $\mathbb{R} \times B(\delta_2)$] for some solution η of the system but $\eta(\tau) \notin W^s$ (respectively W^u), then $\eta(t)$ must leave the ball $B(\delta_1)$ [respectively $B(\delta_2)$] at some finite time $t_1 > \tau$ (respectively $t_2 < \tau$).

These two theorems will be applied to the reference dynamics for $t \leq t_0$ and $t \geq t_f$ during which the reference dynamics becomes the autonomous zero dynamics. With these preparations, we start by first showing that the boundary condition (11) ensures finite energy of the solution of the two-point boundary value problem, but those not satisfying the boundary condition (11) all have infinite energy.

Theorem 3: Suppose that the nonlinear system and the reference trajectory satisfy Assumptions 1 and 2, respectively. Then, among all the solutions $\eta(t)$ of the reference dynamics (10), the $\eta_d(t)$ that satisfies the boundary condition (11) is the only one yielding a finite $\mathcal{L}_2(-\infty, +\infty)$-norm.

Proof: Assumptions 1 and 2 guarantee the existence of a unique $\eta_d(t)$ for all $t \in (-\infty, +\infty)$. Consider

$$\int_{-\infty}^{+\infty} |\eta_d(t)|_2^2 \, dt = \int_{-\infty}^{t_0} |\eta_d(t)|_2^2 \, dt + \int_{t_0}^{t_f} |\eta_d(t)|_2^2 \, dt + \int_{t_f}^{+\infty} |\eta_d(t)|_2^2 \, dt. \quad (12)$$

Since η_d is continuous, it is bounded over a compact interval. Denote

$$K_1 = \sup\{|\eta_d(t)|_2 \mid t_0 \le t \le t_f\}.$$

By the boundary condition (11), $\eta_d(t_f) \in W^s$ for all $t \ge t_f$ since W^s is invariant. By Theorem 1, there exist constants $\alpha_1 > 0$ and $\beta_1 > 0$ such that

$$\begin{aligned}|\eta_d(t)|_2 &\le \alpha_1 |\eta_d(t_f)|_2 \exp\{-\beta_1(t - t_f)\} \\ &\le \alpha_1 K_1 \exp\{-\beta_1(t - t_f)\}, \qquad \forall t \ge t_f.\end{aligned}$$

Similarly, the boundary condition (11) implies that $\eta_d(t_0) \in W^u$ for all $t \le t_0$ and that there exist constants $\alpha_2 > 0$ and $\beta_2 > 0$ such that we have

$$\begin{aligned}|\eta_d(t)|_2 &\le \alpha_2 |\eta_d(t_0)|_2 \exp\{\beta_2(t - t_0)\} \\ &\le \alpha_2 K_1 \exp\{\beta_2(t - t_0)\}, \qquad \forall t \le t_0.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{-\infty}^{t_0} |\eta_d(t)|_2^2 \, dt \le \int_{-\infty}^{t_0} \alpha_2^2 K_1^2 \, e^{2\beta_2(t-t_0)} \, dt = \frac{\alpha_2^2 K_1^2}{2\beta_2}, \\ &\int_{t_0}^{t_f} |\eta_d(t)|_2^2 \, dt \le K_1^2 |t_f - t_0|, \\ &\int_{t_f}^{+\infty} |\eta_d(t)|_2^2 \, dt \le \int_{t_f}^{+\infty} \alpha_1^2 K_1^2 \, e^{-2\beta_1(t-t_f)} \, dt = \frac{\alpha_1^2 K_1^2}{2\beta_1}. \quad (13)\end{aligned}$$

Substituting (13) into (12) we get

$$\int_{-\infty}^{+\infty} |\eta_d(t)|_2^2 \, dt \le M_1^2 < +\infty \quad (14)$$

where the constant

$$M_1 \stackrel{\text{def}}{=} \left[\frac{\alpha_2^2 K_1^2}{2\beta_2} + K_1^2 |t_f - t_0| + \frac{\alpha_1^2 K_1^2}{2\beta_1}\right]^{1/2}$$

that is

$$\|\eta_d\|_{\mathcal{L}_2(-\infty, +\infty)} = \left[\int_{-\infty}^{+\infty} |\eta_d(t)|_2^2 \, dt\right]^{1/2} \le M_1 < +\infty. \quad (15)$$

On the other hand, consider any other solution $\eta(t)$ of (10) that does not satisfy the boundary condition (11), that is

$$\eta(t_0) \notin W^u \quad \text{and/or} \quad \eta(t_f) \notin W^s.$$

Suppose $\eta(t_f) \notin W^s$, then $\eta(t) \notin W^s$ for all $t \ge t_f$ due to the invariance of W^s. We want to show that the $\mathcal{L}_2(-\infty, +\infty)$-norm of this solution is infinite by showing

$$\int_{t_f}^{+\infty} |\eta(t)|_2^2 \, dt = +\infty. \quad (16)$$

Select a constant $\delta_1 = 2\delta > 0$ as in Theorem 2. Without loss of generality, we assume that $|\eta| = 2\delta$ is not an equilibrium, since otherwise we immediately have (16). Let $\{t_k, \, k = 1, 2, 3, \cdots, t_{k+1} > t_k \ge t_f\}$ be the set of all time points at which η enters the ball. If this set is empty and $\eta(t_f) \in B(2\delta)$, η will leave the ball in finite time and stay outside for the rest of the time, or if $\eta(t_f) \notin B(2\delta)$ it will remain outside the ball for all $t \ge t_f$. In either case, (16) is obviously true. If the set is nonempty, we construct a new set $\{t'_k, \, k = 1, 2, 3, \cdots, t'_{k+1} > t'_k \ge t_f\}$ as follows. Let $t'_1 = t_1$ and $I_1 = [t'_1, t'_1 + \Delta t]$. Then find the first $t_j \notin I_1$ in the t_k set and let $t'_2 = t_j$ and $I_2 = [t'_2, t'_2 + \Delta t]$. Continue this process until the t_k set is exhausted. The constant Δt in this process is defined by

$$\Delta t = \frac{\delta}{\max |p[y_d^{(r)}, \, \xi_d, \eta \in B(2\delta)]|}.$$

With this Δt, it follows easily that $|\eta(t)| \ge \delta$ for all $t \in I_k$ since $|\eta(t'_k)| = 2\delta$. Also notice that all these I_k's are disjoint.

Two situations need to be considered. First, the set t'_k contains a finite number of points. Then by Theorem 2, $\eta(t)$ will leave the ball in finite time after each I_k. Therefore, the total amount of time during which $\eta(t)$ is inside the ball is finite, and during the rest of the time it is outside the ball or $|\eta(t)| > 2\delta$. Consequently, (16) is true.

In the second situation, the set $\{t'_k\}$ contains an infinite number of points. In this case, we have

$$\begin{aligned}\int_{t_f}^{+\infty} |\eta(t)|_2^2 \, dt &\ge \int_{\cup_{k=1}^{\infty} I_k} |\eta(t)|_2^2 \, dt \ge \sum_{k=1}^{\infty} \int_{I_k} \delta^2 \, dt \\ &= \sum_{k=1}^{\infty} \delta^2 \Delta t\end{aligned}$$

which is unbounded and implies (16).

A similar argument can be applied when $\eta(t_0) \notin W^u$. Hence, violating any part of the boundary condition (10) always leads to $\|\eta\|_{\mathcal{L}_2(-\infty, +\infty)} = \infty$. □

The following two theorems claim that both the desired state trajectory x_d and the nominal control input u_d obtained via $\eta_d(t)$ have finite $\mathcal{L}_2(-\infty, +\infty)$-norms, while those obtained via any other $\eta(t)$ have infinite $\mathcal{L}_2(-\infty, +\infty)$-norms. Due to the limitation on paper size, both proofs are omitted.

Theorem 4: Suppose that the nonlinear system and the reference trajectory satisfy Assumptions 1 and 2, respectively. Then, among the infinitely many state trajectories $x(t)$ which map exactly into the desired output trajectory, the $x_d(t)$ computed by $x_d = \Phi^{-1}(\xi_d, \, \eta_d)$, where η_d is the solution of (10) subjected to (11), is the only one yielding a finite $\mathcal{L}_2(-\infty, +\infty)$-norm.

To establish the next theorem on the property of the nominal control input, we pose the following technical assumption that is used in the proof of the theorem.

Assumption 3: On the zero dynamics manifold $\xi = 0$:

1) $L_g L_f^{r-1} h(x)$ is globally uniformly bounded;
2) given any $\delta > 0$, $\Delta t > 0$, there exists an $\epsilon(\Delta t, \, \delta) > 0$ such that for all t, $|\eta(\tau)| > \delta$ for all $\tau \in [t, \, t + \Delta t]$ implies that $\|\alpha(0, \, \eta)\|_{\mathcal{L}_2[t, t+\Delta t]}^2 \ge \epsilon$.

It is noticed that the first condition in the assumption does not pose any practical constraints since matrix β is the high-frequency gain from input to output which is bounded for any practical systems. The second one is related to an observability condition. In the linear case, if the zero dynamics is observable from α, then the condition is satisfied.

Theorem 5: Suppose that the nonlinear system and the reference trajectory satisfy Assumptions 1–3. Then, among all the control inputs which reproduce exactly the reference trajectory, the u_d computed by $u_d = [\beta(\xi_d, \, \eta_d)]^{-1}[y_d^{(r)} - \alpha(\xi_d, \, \eta_d)]$, where η_d is the solution of (10) subjected to (11), is the only one yielding a finite $\mathcal{L}_2(-\infty, +\infty)$-norm.

Remark: In practice, it is impossible to work with the interval $(-\infty, +\infty)$. Instead, $[-T, \, T]$ will be used where $-T \le t_0$ and $T \ge t_f$. Suppose $u(t)$ is any control input defined on $[-T, \, T]$ and it produces $y(t) \equiv y_d(t)$ on $[-T, \, T]$. Then we can claim, as an immediate consequence of Theorem 5, that $\|u(t)\|_{\mathcal{L}_2[-T, \, T]}$

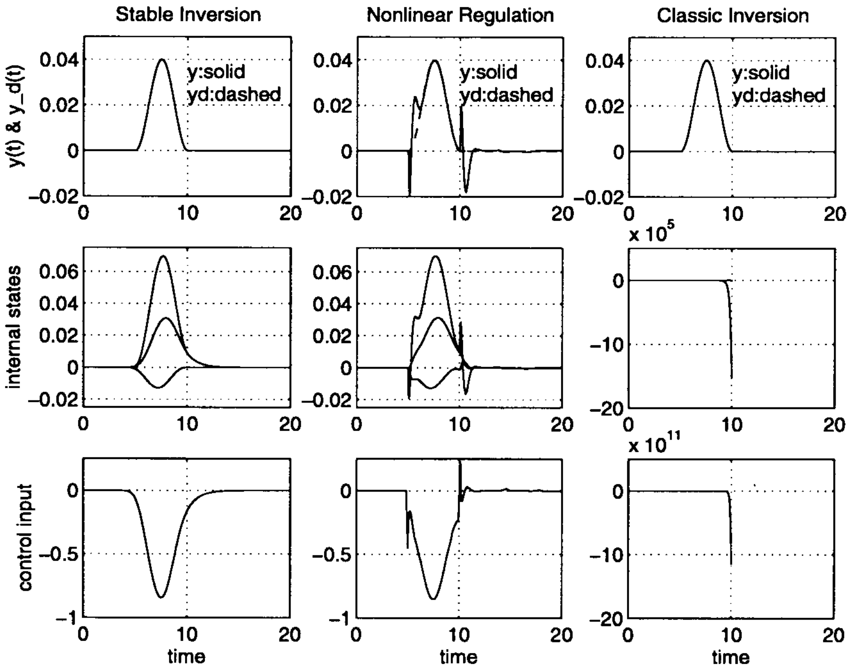

Fig. 1. Simulation results by three output tracking approaches.

is greater than $\| u_d(t) \|_{\mathcal{L}_2[-T,\,T]}$ for T sufficiently large. In this sense, $u_d(t)$ is the minimum energy input among all exact-output-reproducing input signals. Note that we did not explicitly specify the initial condition $\eta(-T)$ since it has a one-to-one relationship with $u(t)$ and therefore is implicitly specified. Similar comments can also be made regarding $\eta_d(t)$ and $x_d(t)$.

On the other hand, it should be pointed out that it is possible to achieve approximate output tracking with an appropriate $u(t)$ that has smaller energy than $u_d(t)$ on $[-T,\,T]$ if exact output reproduction is not required. This is easy to conceive since $u_d(t)$ is not optimized for energy on $[-T,\,T]$.

IV. An Example

The effectiveness of stable inversion in output tracking and its properties established in the previous section can be illustrated by the following example of a nonlinear single-input/single-output system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3x_2 + x_2^2 + x_3^2 \\ x_1 + 3x_2 - x_3 \\ -3x_2 - x_3 - x_2^2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u \tag{17}$$

$$y = x_1 - x_3 \tag{18}$$

with a reference output trajectory given by

$$y_d = \begin{cases} 0.02 - 0.02\,\cos(0.4\pi[t-5]), & t \in [5,\,10] \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

It can be easily verified that all three assumptions required to apply stable inversion are satisfied. By applying a change of coordinates

$$\begin{cases} \xi = x_1 - x_3 \\ \eta_1 = x_2 \\ \eta_2 = x_3 \end{cases}$$

and letting $\xi = y = y_d$, the desired output, the reference dynamics can be derived to be

$$\dot{\eta}_1 = 3\eta_1 + y_d \tag{20}$$

$$\dot{\eta}_2 = -3\eta_1 - \eta_2 - \eta_1^2. \tag{21}$$

The stable manifold of the zero dynamics is characterized by $\eta_1 = 0$, and the unstable manifold by

$$\eta_2 = -\tfrac{3}{4}\eta_1 - \tfrac{1}{7}\eta_1^2. \tag{22}$$

Based on the results obtained in last section, there is only one trajectory satisfying the above reference dynamics yielding a finite $\mathcal{L}_2(-\infty,\,+\infty)$-norm, and it is the $\eta_d(t)$ given by stable inversion. Thus, any numerical iterative algorithm can be used to compute $\eta_d(t)$ as long as it converges and produces a finite $\mathcal{L}_2(-\infty,\,+\infty)$-norm solution. The method chosen here is as follows: first, integrate the unstable part of the reference dynamics that is given by (20) backward in time with zero final condition to obtain $\eta_1(t)$; second, use (22) for the unstable manifold to obtain the required initial condition for η_2; then, integrate the stable part of the reference dynamics in (21) forward in time to obtain $\eta_2(t)$.

Simulation results are shown in the first column of Fig. 1. In this simulation a linear stabilizing feedback is implemented which gives the following closed-loop dynamics:

$$\dot{x} = f(x) + g(x)u = f(x) + g(x)\{u_d + k[x - x_d]\}, \qquad x(0) = 0$$

where $k = [-7.5,\,-50,\,25]$ is the feedback gain. It can be seen from the plots that a remarkably accurate output reproduction is achieved, and the internal dynamics of the system is stable with desired state trajectories and the nominal control input approach zero as time goes to either plus or minus infinity.

As a comparison, let us first consider the nonlinear regulator approach. A rest initial condition is also assumed. The reference trajectory can be generated by the following exosystem:

$$\begin{cases} \dot{w}_1 = \dfrac{2\pi}{5}\,w_2 \\ \dot{w}_2 = -\dfrac{2\pi}{5}\,w_1 \\ \dot{w}_3 = 0 \end{cases}$$

$$y_d = w_3 - w_1$$

TABLE I
A COMPARISON ON CONTROL AND STATES ENERGY AND TRACKING ERRORS

Approaches	Stable Inversion $I = [0, 20]$	Nonlinear Regulation $I = [0, 20]$	Classic Inversion $I = [0, 10]$
$\| u \|_{\mathcal{L}_2(I)}$	1.2632	1.3158	2.1395×10^{11}
$\| x \|_{\mathcal{L}_2(I)}$	0.1112	0.1161	5.9425×10^{5}
$\frac{\| y - y_d \|_{\mathcal{L}_2(I)}}{\| y_d \|_{\mathcal{L}_2(I)}}$	6.3388×10^{-4}	0.3493	5.0094×10^{-4}
$\frac{\| y - y_d \|_{\mathcal{L}_\infty(I)}}{\| y_d \|_{\mathcal{L}_\infty(I)}}$	8.2342×10^{-4}	0.4886	8.5586×10^{-4}

with initial conditions being set and reset as follows:

$$\begin{cases} w_1(-\infty) = w_2(-\infty) = w_3(-\infty) = 0 \\ w_1(5) = w_3(5) = 0.02; \qquad w_2(5) = 0 \\ w_1(10) = w_2(10) = w_3(10) = 0. \end{cases}$$

After solving a set of partial differential equations, we obtain the zero-error manifold, $x^* = \pi(w)$ and $u^* = u^*(w)$, in a closed form, and the closed-loop dynamics, which shares the same feed-forward plus feedback structure and the same linear feedback gain k with the stable inversion implementation, is given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ &= f(x) + g(x)\{u^*(w) + k[x - \pi(w)]\} \\ x(0) &= 0. \end{aligned}$$

Simulation results by this regulator design are shown in the second column of Fig. 1 from which it can be seen that the asymptotic tracking is achieved with transient errors existing both at the beginning and at the end of the output maneuver. Comparing stable inversion and nonlinear regulation, it is noticed that both approaches have a mismatched initial condition at $t = 0$ and both assure asymptotic convergence of output errors. However, because of the noncausal or preview characteristic of stable inversion, it is capable of transition without causing transient output errors. In contrast, the nonlinear regulation approach does not have such capability.

In the third column of Fig. 1 are those plots of simulation results by the classic inversion approach. The unboundedness of internal states and control input shown is due to the unstable zero dynamics.

The control energy, internal states energy, and norms of tracking errors corresponding to three different approaches are summarized in Table I.

Norms corresponding to stable inversion and nonlinear regulation are computed over the time interval $[0, 20]$ since all the values considered are essentially zero before $t = 0$ and after $t = 20$. Hence, those norms can be thought of as the norms on $(-\infty, +\infty)$. Norms corresponding to the classic inversion approach are computed over the interval $[0, 10]$. For the output, these norms are the same as those calculated on $(-\infty, +\infty)$ since $y \equiv y_d \equiv 0$ for $t \leq 0$ and $t \geq 10$. However, the norms of u and x become infinity on $(-\infty, +\infty)$. It is clearly seen from the table the advantages of stable inversion over the other two approaches. As mentioned earlier, the price stable inversion pays is that these inverse signals are noncausal and computed offline. However, online implementation is also possible, provided that the reference output trajectory is known beforehand, since "noncausal" implementation here actually means computing by using "anticipatory" reference data.

V. CONCLUSION

For nonlinear systems of the form (1) and (2) with a well-defined relative degree and hyperbolic zero dynamics, the stable inversion problem is guaranteed to have a unique solution under mild conditions on the reference output trajectory. In this paper, we have shown that the stable inverse solution enjoys some very important properties beyond the guaranteed boundedness and convergence, even if the system has nonminimum phase. Specifically, the nominal control input is the only one that uses a finite amount of energy to reproduce exactly the reference trajectory and at the same time to cause a finite amount of internal vibrations. The stable inverse solution can be used in tracking controllers that guarantee closed-loop stability and exact output tracking without any transients. Using the same structure for the forward feedback controller as that used by nonlinear regulation, stable inversion is also a model-based approach and has the same robustness issue as nonlinear regulation does. To handle the model uncertainty, either a robust feedback controller needs to be designed or some robust stable inverse solutions be computed. The robustness issue of inverse solutions is an issue of stable inversion and is currently under study.

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