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Convergence in the presence of zero-ergodic-mean disturbance

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This paper considers the convergence of systems subjected to zero-ergodic-mean disturbance. We first consider the stability of averaged systems and give an example showing that exponential stability does not imply averaged stability. Next a sufficient condition is established for a class of averaged systems to be stable. Using this result, we obtain conditions guaranteeing convergence of systems subjected to zero-ergodic-mean disturbance.

Keywords: stability; averaged systems; convergence; disturbance.

1. Introduction

Robustness to disturbance is an important issue which has attracted much attention. However, general results tend to give conservative bounds on the tolerable disturbance. On the other hand, disturbance in practice is usually structured, making better bounds possible. In a recent study (Chen & Paden 1990, 1993) on parameter convergence in a nonlinear adaptive control of torque-ripple cancellation for permanent-magnetic step motors, one such example is encountered. In this case, the disturbance is the residual torque-ripple not cancelled by an adaptive loop. Its absolute value may be very large, but it is roughly sinusoidal and has zero ergodic mean (Wiener 1964). Available results (e.g. Sastry & Bodson 1989) on bounded error with bounded disturbance lead to a bound on the parameter error that is large, since the bound on the disturbance is large. Nevertheless, experimental data indicate parameter convergence in the presence of such disturbances.

Motivated by the above, this paper studies the convergence properties of a class of systems subjected to zero-ergodic-mean disturbance. Typically, a controller is designed based on an unperturbed model, resulting in an exponentially stable closed-loop system in the absence of disturbance. Intuitively, one would expect that the system dynamics will smooth out the zero-ergodic-mean disturbance, rendering the averaged value of state error small. This is indeed the case under certain conditions, and can be proved using a converse averaging theorem.

The averaging method has long been used to study nonlinear dynamical systems (Sanders & Bodson 1985) and ordinary differential equations (Hale 1969). In more recent years, this method has been successfully used in the study of parameter convergence

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in adaptive control systems (e.g. Mason *et al.* 1987; Kosut *et al.* 1987) and nonlinear oscillators (Grace & Lalli 1990). In control systems, the stability of averaged systems is used to establish that of the un-averaged systems. A typical such result states that, under certain technical conditions, if the averaged system is exponentially stable, then the original system is also exponentially stable.

The question posed by converse averaging is: given that the original system is exponentially stable, when is the averaged system also stable? The answer to this seemingly trivial question is unfortunately not 'always', as shown by the example in Section 2. Section 3, however, does establish a sufficient condition for the averaged systems to be exponentially stable. Finally, Section 4 studies a class of exponentially stable systems subjected to zero-mean disturbances, and obtains a condition for the states to converge on the average to a small neighbourhood of zero.

In this paper, we will use the following convention for norms of matrices (including vectors) and signals. If $a \in \mathbb{R}^{n\times m}$, then $|a|$ is the largest singular value of a. If $a : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, then $\|a\| = \sup_{t\in\mathbb{R}_+} |a(t)|$. In addition, if $a \in \mathbb{R}^{n\times n}$, then $\lambda_i(a)$ is the ith eigenvalue of a (in an arbitrary listing).

2. A stable system whose average is unstable

Consider a time-dependent second-order system given in polar coordinates:

$$\dot r = r[1 - 3\cos(\phi - t)], \qquad \dot\phi = 2 - 2\cos(\phi - t). \tag{2.1}$$

Averaging this system with respect to time for a period 2π yields

$$\dot r = r, \qquad \dot\phi = 1$$

which is clearly unstable. However, the system (2.1) can be shown stable as follows. First, make a change of variable $\theta = \phi - t$ to achieve

$$\dot r = r[1 - 3\cos\theta], \qquad \dot\theta = 1 - 2\cos\theta. \tag{2.2}$$

There are two equilibria for θ at $\pm\cos^{-1}\frac{1}{2}$. It is then straightforward to conclude that θ converges to the stable equilibrium $-\cos^{-1}\frac{1}{2}$. Finally, since the vector field in (2.2) is continuous, and since

$$\dot r = -\tfrac{1}{2}r$$

at the equilibria for θ, we have that the origin is stable for the system (2.2). The same is true for system (2.1). Therefore stability of the original system does not imply averaged stability in general.

Figure 1 shows the phase portrait of the system (2.1), drawn in cartesian coordinates. Six trajectories with different initial conditions are shown. It is clear that (0,0) is an asymptotically stable equilibrium point of the original system. Notice that the trajectories cross each other. This is due to the fact that the system is non-autonomous. For a time-invariant system, trajectories never cross each other.

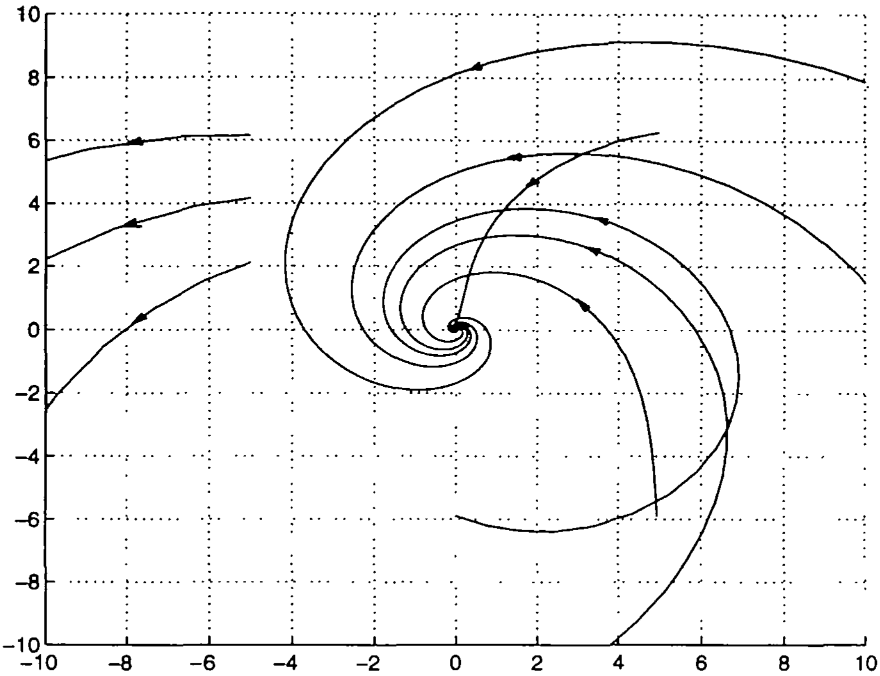

FIG. 1. Phase portrait in cartesian coordinates

3. Stability of averaged systems

Let $A(t) \in \mathbf{L}_{\infty}^{n\times n}$ be bounded and piecewise continuous, and

$$B(t) = \frac{1}{T}\int_t^{t+T} A(\tau)\,\mathrm{d}\tau, \tag{3.1}$$

where the averaging period T is a free parameter. Define two systems:

$$\dot{x}(t) = \gamma A(t)x(t), \tag{3.2}$$

$$\dot{y}(t) = \gamma B(t)y(t), \tag{3.3}$$

where γ is another free parameter. In this section, we consider the stability of (3.3) given that (3.2) is exponentially stable. The development starts with the following lemma which puts a bound on the growth rate of the state transition matrix.

LEMMA 1 Let $\Phi(t, t_0)$ be the state transition matrix of (3.2). Then, for $t > t_0 \geqslant 0$,

$$|\Phi(t,t_0) - I| \leqslant \exp(\gamma\|A\|(t-t_0)) - 1. \tag{3.4}$$

Proof. First note that

$$\Phi(t,t_0) - I = \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}s}\Phi(s,t_0)\,\mathrm{d}s = \int_{t_0}^{t} \gamma A(s)\Phi(s,t_0)\,\mathrm{d}s$$

$$= \int_{t_0}^{t} \{\gamma A(s) + \gamma A(s)(\Phi(s,t_0) - I)\}\,\mathrm{d}s.$$

Therefore

$$\begin{aligned}|\Phi(t,t_0) - I| &\leqslant \int_{t_0}^{t} \{\gamma|A(s)| + \gamma|A(s)|\,|(\Phi(s,t_0) - I)|\}\,\mathrm{d}s \\ &\leqslant \gamma(t-t_0)\|A\| + \gamma\|A\| \int_{t_0}^{t} |(\Phi(s,t_0) - I)|\,\mathrm{d}s.\end{aligned}$$

Use of the Gronwall–Bellman lemma in the appendix with ν constant, and setting $h(t) = \gamma(t-t_0)\|A\|$, leads to

$$|\Phi(t,t_0) - I| \leqslant \gamma\|A\| \int_{t_0}^{t} \exp(\gamma\|A\|(t-s))\,\mathrm{d}s = \exp(\gamma\|A\|(t-t_0)) - 1.$$

□

COROLLARY 1 Let $x(t)$ be the solution to (3.2). For $t > \tau \geqslant 0$,

$$|x(t) - x(\tau)| \leqslant |x(\tau)|[\exp(\gamma\|A\|(t-\tau)) - 1].$$

The next lemma provides an approximation formula for the state transition matrix which is useful to both averaging and converse averaging.

LEMMA 2 Let Φ be the state transition matrix of (3.2). Then

$$\Phi(t+T,t) = I + \gamma T B(t) + (\gamma T)^2 M(t) \tag{3.5}$$

for some M satisfying

$$|M(t)| \leqslant \tfrac{1}{2}\|A\|^2(1 + O(\gamma T\|A\|)).$$

Proof. Integrating (3.2) starting from an arbitrary $x(t)$ at time t to time $t+T$ gives

$$\begin{aligned}x(t+T) - x(t) &= \int_{t}^{t+T} \gamma A(\tau)x(\tau)\,\mathrm{d}\tau \\ &= \int_{t}^{t+T} [\gamma A(\tau)x(t) + \gamma A(\tau)(x(\tau) - x(t))]\,\mathrm{d}\tau.\end{aligned}$$

Hence, by the definition of the state transition matrix, we have

$$\Phi(t+T,t)x(t) = x(t) + \gamma T\left(\frac{1}{T}\int_{t}^{t+T} \gamma A(\tau)\,\mathrm{d}\tau\right)x(t) + \gamma\left(\int_{t}^{t+T} A(\tau)(\Phi(\tau,t) - I)\,\mathrm{d}\tau\right)x(t).$$

Therefore, since $x(t)$ is arbitrary, (3.5) is true with

$$M(t) = \int_{t}^{t+T} \frac{1}{\gamma T^2} A(\tau)(\phi(\tau,t) - I)\,\mathrm{d}\tau.$$

The bound for M is calculated by using Lemma 1:

$$|M(t)| \leqslant \int_t^{t+T} \frac{1}{\gamma T^2} \|A\| (\exp(\gamma \|A\| (\tau - t) - 1) \, d\tau$$
$$= \frac{1}{\gamma^2 T^2} \{(\exp(\gamma T \|A\|) - 1) - \gamma T \|A\|\}.$$

Expanding the exponential function completes the proof. □

A similar result is given by Anderson *et al.* (1986) using the contraction-mapping theorem. However, due to the contraction requirement, their proof is valid only for the case, roughly speaking, when $\gamma T \|A\| < 1$ strictly.

THEOREM 1 let Φ be the state transition matrix of (3.2). If (3.2) is exponentially stable and $|\Phi(t, \tau)| \leqslant m e^{-\lambda(t-\tau)}$ for $t \geqslant \tau \geqslant 0$ with $\lambda = \gamma k'$ and $m = e^{\lambda k''}$ for some $k', k'' > 0$, then there exists $T^* > 0$ such that, for all $T > T^*$, for some $\gamma^*(T)$, and for all $\gamma \in (0, \gamma^*)$, (3.3) is exponentially stable with rate $\gamma k_1 + O(\gamma^2)$ for some $k_1 > 0$.

Proof. By assumption, we have

$$|\Phi(t + T, t)| \leqslant m e^{-\lambda T} = e^{-\gamma (k' T - k'')}.$$

Now we want to establish the exponential stability of (3.3) by using Lemma S in the appendix. From (3.5),

$$\gamma B(t) = \frac{1}{T}(-I + \Phi(t + T, t) - O(\gamma^2 T^2) M),$$

and we immediately have

$$\lambda_i(\gamma B(t)) = \frac{1}{T} \lambda_i (-I + \Phi(t + T, t) - O(\gamma^2 T^2) M).$$

Recalling the continuity of eigenvalues and allowing γ to be small, we have

$$\begin{aligned}
\operatorname{Re}(\lambda_i(\gamma B(t))) &\leqslant \frac{1}{T}(-1 + |\Phi(t + T, t)| + (\gamma T)^2 |M|) \\
&\leqslant \frac{1}{T}(-1 + e^{-\gamma T (k' - k''/T)} + (\gamma T)^2 |M|) \\
&\leqslant \frac{1}{T}\left[-\gamma T \left(k' - \frac{k''}{T}\right) + \frac{1}{2}(\gamma T)^2 \left(k' - \frac{k''}{T}\right)^2 + (\gamma T)^2 |M| \right] \\
&= -\gamma \left(k' - \frac{k''}{T}\right) + \gamma^2 T \delta =: -\lambda(\gamma),
\end{aligned}$$

where $\delta > 0$ is defined in an obvious way and is clearly bounded. On the other hand,

$$\frac{d}{dt}(\gamma B(t)) = \gamma \frac{1}{T}(A(t + T) - A(t)).$$

Since $A(t)$ is bounded ($b := 2\|A\|$),

$$\left| \frac{d}{dt}(\gamma B(t)) \right| \leqslant \gamma \frac{1}{T} b.$$

Hence, letting ε be as in Lemma S, we require $\lambda(\gamma) > 0$ and $\gamma T^{-1}b \leqslant \varepsilon\lambda(\gamma)$, which reduces to

$$0 < \gamma T < \left(k' - \frac{k''}{T}\right) \Big/ \delta,$$

$$\frac{b}{T} < \varepsilon < k' - \frac{k''}{T} - \gamma T \delta.$$

Equating the two sides to solve for γ^* and setting γ to zero to solve for T^*, we get

$$T^* = \frac{b}{\varepsilon k'} + \frac{k''}{k'},$$

$$\gamma^*(T) = \left(k' - \frac{k''}{T} - \frac{b}{\varepsilon T}\right) \Big/ \delta T.$$

Then $T > T^*$ and $\gamma \in (0, \gamma^*(T))$ will ensure the exponential stability of (3.3). Furthermore, it can be easily seen that the convergence rate, once T is chosen, is $\gamma k_1 + O(\gamma^2)$, where $k_1 > 0$ is some constant dependent on T. □

REMARK 1 This result is restricted in the sense that we put a special requirement on the form of the overshoot m. However, systems such as the parameter-adaptation loop in many adaptive control systems satisfy this. Furthermore, our assumption on the A matrix is fairly weak.

4. Convergence of perturbed systems

In this section, we consider system (3.2) subjected to a perturbation input:

$$\dot{x} = \gamma A(t) + \gamma g(t), \tag{4.1}$$

where $g(t) \in \mathbf{L}_\infty^n$ has zero ergodic mean; that is,

$$\lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} g(\tau)\,\mathrm{d}\tau = 0. \tag{4.2}$$

In the nonlinear adaptive torque-ripple cancellation for step motors (Chen & Paden 1990, 1993), the parameter-update loop looks similar to this. The $g(t)$ is the residual torque-ripple harmonics not cancelled by the control. It is roughly sinusoidal with zero average. Experimental results indicate that by slowing down (reducing γ) the adaptation, parameter errors can be smoothed and reduced.

The question we ask here is: in the system (4.1), can the averaged value of x (and x itself) be made to converge to zero? The answer, under certain conditions, is yes. We first establish the following result which is similar to the corollary to Lemma 1.

LEMMA 3 Let $x(t)$ be a solution of system (4.1). Then

$$|x(\tau) - x(t)| \leqslant \gamma(\|A\|\,|x(t)| + \|g\|)|\tau - t| + O(\gamma^2(\tau - t)^2). \tag{4.3}$$

Proof. Without loss of generality, we assume $\tau \geqslant t$. From equation (4.1), we have

$$
\begin{aligned}
|x(\tau) - x(t)| &= \left| \int_t^\tau (\gamma A(s)x(s) + \gamma g(s))\,\mathrm{d}s \right| \\
&\leqslant \int_t^\tau |\gamma A(s)x(t) + \gamma g(s) + \gamma A(s)(x(s) - x(t))|\,\mathrm{d}s \\
&\leqslant \gamma \int_t^\tau (|A(s)|\,|x(t)| + |g(s)|)\,\mathrm{d}s + \int_t^\tau \gamma |A(s)|\,|x(s) - x(t)|\,\mathrm{d}s \\
&\leqslant \gamma(\tau - t)\left(\|A\|\,|x(t)| + \|g\| + \int_t^\tau \gamma |A(s)|\,|x(s) - x(t)|\,\mathrm{d}s \right).
\end{aligned}
$$

The Gronwall–Bellman lemma with ν constant and $h(\tau) = \gamma(\tau - t)(\|A\|\,|x(t)| + \|g\|)$ gives

$$
\begin{aligned}
|x(\tau) - x(t)| &\leqslant \gamma(\|A\|\,|x(t)| + \|g\|) \int_t^\tau \exp(\gamma \|A\|(\tau - s))\,\mathrm{d}s \\
&\leqslant \left(|x(t)| + \frac{\|g\|}{\|A\|} \right) (\exp(\gamma \|A\|(\tau - t)) - 1).
\end{aligned}
$$

Expanding the exponential function proves the lemma. □

THEOREM 2 Let the conditions in Theorem 1 be satisfied, and let $g(t)$ have zero ergodic mean. Let $x(t)$ be a solution of (4.1), and define its averaged value by

$$
y(t) = \frac{1}{T} \int_t^{t+T} x(\tau)\,\mathrm{d}\tau.
$$

Then y converges to a neighbourhood of zero whose size is arbitrarily small for sufficiently large T and sufficiently small γ.

Proof. Differentiating y gives

$$
\dot{y} = \frac{\gamma}{T} \int_t^{t+T} A(\tau)x(\tau)\,\mathrm{d}\tau + \frac{\gamma}{T} \int_t^{t+T} g(\tau)\,\mathrm{d}\tau.
$$

By Theorem 1, the averaged system $\dot{y} = \gamma B(t)y$ is exponentially stable with sufficiently large T and sufficiently small γ. We separate this stable part in the above equation and call the rest the *averaged perturbation*. Then we put a bound on the averaged perturbation and choose appropriate T and γ to make it small. Following this idea, we have

$$
\dot{y} = \gamma B(t)y + \frac{\gamma}{T} \int_t^{t+T} g(\tau)\,\mathrm{d}\tau + \frac{\gamma}{T} \int_t^{t+T} A(\tau)(x(\tau) - y(t)\,\mathrm{d}\tau. \tag{4.4}
$$

The second term on the right-hand side of (4.4) can be made arbitrarily small due to (4.2), i.e. for a given $\varepsilon > 0$, there exists $T^{\#}$ such that $T \geqslant T^{\#}$ implies that

$$
\left| \frac{1}{T} \int_t^{t+T} g(\tau)\,\mathrm{d}\tau \right| \leqslant \varepsilon.
$$

Hence, for such T, the second term is bounded by $\gamma\varepsilon$. Now, to bound the third term, first let $z(\tau) = x(\tau) - y(t)$. Then, since $\int_t^{t+T} z(\tau)\,d\tau = 0$, there exists $\xi \in [t, t+T]$ such that $z(\xi) = 0$. Next, note

$$\dot{z}(\tau) = \gamma A(\tau)x(\tau) + \gamma g(\tau) = \gamma A(\tau)z(\tau) + \gamma(A(\tau)y(t) + g(\tau)).$$

Applying Lemma 3 to this system leads to

$$\begin{aligned}|z(\tau)| &= |z(\tau) - z(\xi)| \\ &\leqslant \gamma(\|A\|\,|z(\xi)| + \|A\|\,y(t)| + \|g\|)|\tau - \xi| + O(\gamma^2(\tau - \xi)^2) \\ &= \gamma(\|A\|\,|y(t)| + \|g\|)|\tau - \xi| + O(\gamma^2(\tau - \xi)^2).\end{aligned} \tag{4.5}$$

Finally, the third term in the right-hand side of (4.4) can be bounded:

$$\begin{aligned}\left|\frac{\gamma}{T}\int_t^{t+T} A(\tau)(x(\tau) - y(t))\,d\tau\right| &\leqslant \frac{\gamma}{T}\int_t^{t+T} |A(\tau)|\,|x(\tau) - y(t)|\,d\tau \\ &\leqslant \frac{\gamma}{T}\|A\|\int_t^{t+T} |z(\tau)|\,d\tau \\ &\leqslant \frac{\gamma}{T}\|A\|\int_t^{t+T} [\gamma(\|A\|\,|y(t)| + \|g\|)|\tau - \xi| + O(\gamma^2(\tau - \xi)^2)]\,dt \\ &\leqslant \tfrac{1}{2}\gamma^2 T\|A\|(\|A\|\,|y(t)| + \|g\|) + O(\gamma^3 T^2).\end{aligned}$$

With these preparations, let us now return to (4.4). By Theorem 1, when T is sufficiently large and γ is sufficiently small, $\dot{y} = \gamma B(t)y(t)$ is exponentially stable with convergence rate $\gamma k_1 + O(\gamma^2)$. By the converse Lyapunov theorem, there exist a function $V(y(t), t)$ and constants α_1, α_2, and α_3 such that

$$\alpha_1|y(t)|^2 \leqslant V \leqslant \alpha_2|y(t)|^2,$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial y}\gamma B(t)y(t) \leqslant -(\gamma k_1 + O(\gamma^2))|y(t)|^2, \qquad \left|\frac{\partial}{\partial y}V\right| \leqslant \alpha_3|y(t)|.$$

Using this V as a Lyapunov function candidate and differentiating it along the solution of (4.4) leads to

$$\begin{aligned}\frac{d}{dt}V\bigg|_{(4.4)} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial y}\gamma B(t)y(t) + \frac{\partial}{\partial y}V\frac{\gamma}{T}\int_t^{t+T} g(\tau)\,d\tau \\ &\quad + \frac{\partial}{\partial y}V\frac{\gamma}{T}\int_t^{t+T} A(\tau)(x(\tau) - y(t))\,d\tau.\end{aligned}$$

Using the bounds we have established results in

$$\begin{aligned}\frac{d}{dt}V\bigg|_{(4.4)} &\leqslant -(\gamma k_1 + O(\gamma^2))|y(t)|^2 + \alpha_3\gamma\varepsilon|y(t)| \\ &\quad + \tfrac{1}{2}\alpha_3|y(t)|\gamma^2 T\|A\|(\|A\|\,|y(t)| + \|g\|) + O(\gamma^3 T^2) \\ &= -(\gamma k_1 + O(\gamma^2) - \tfrac{1}{2}\alpha_3\gamma^2 T\|A\|^2)|y(t)|^2 \\ &\quad + \alpha_3\gamma|y(t)|(\varepsilon + \tfrac{1}{2}\gamma T\|A\|\,\|g\| + O(\gamma^2 T^2)) \\ &= -\gamma|y(t)|\{k_2|y(t)| - \alpha_3\varepsilon - \tfrac{1}{2}\alpha_3\gamma T\|A\|\,\|g\| - O(\gamma^2 T^2)\},\end{aligned}$$

where $k_2 = k_1 - O(\gamma T)$ is defined in an obvious way. It is clear that, when γ and γT are sufficiently small, k_2 is positive. Hence, as $t \to \infty$, $y(t)$ converges to a neighbourhood of zero, the size of which is

$$\frac{\alpha_3\varepsilon + \frac{1}{2}\alpha_3\gamma T\|A\|\,\|g\| + O(\gamma^2T^2)}{k_2}.$$

Note that the first term on the numerator can be made arbitrarily small by choice of large T, and the rest of the numerator can be made as small as we please by choice of small enough γ, while the denominator is relatively constant in such a process. This shows that the steady-state value of y can be made arbitrarily small by choice of large T and small γ. □

REMARK 2 If the disturbance input $g(t)$ is periodic or nearly periodic, T can be selected to be integral multiples of its period, so that ε can be made very small with T not necessarily very large. Such situations happen quite frequently in practical applications.

THEOREM 3 Under the conditions of Theorem 2, $x(t)$ converges to 0 as $\gamma \to 0$.

Proof. Simply note that, from (4.5), $|x(t) - y(t)|$ is bounded by a term proportional to γT. □

COROLLARY 2 If $g(t)$ is stationary and there is a constant vector G such that

$$\lim_{T\to\infty}\left|\frac{1}{T}\int_t^{t+T}(g(\tau) - G)\,\mathrm{d}\tau\right| = 0,$$

Then x will converge to a small neighbourhood of the constant steady-state solution of

$$\dot{a} = \gamma A(t)a + \gamma G$$

for sufficiently small γ.

Proof. Perform a change of variable, $z = x - a$, and apply Theorem 3 to the z dynamics. □

COROLLARY 3 If there exists a function $G(t)$ whose difference from $g(t)$ has stationary zero ergodic mean, i.e.

$$\lim_{T\to\infty}\left|\frac{1}{T}\int_t^{t+T}(g(\tau) - G(t))\,\mathrm{d}\tau\right| = 0 \quad (t \geqslant 0),$$

then x will converge to an arbitrarily small neighbourhood of the solution of

$$\dot{a} = \gamma A(t)a + \gamma G(t)$$

for sufficiently small γ.

Proof. Again, perform a change of variable, $z = x - a$, and apply Theorem 3 to the z dynamics. □

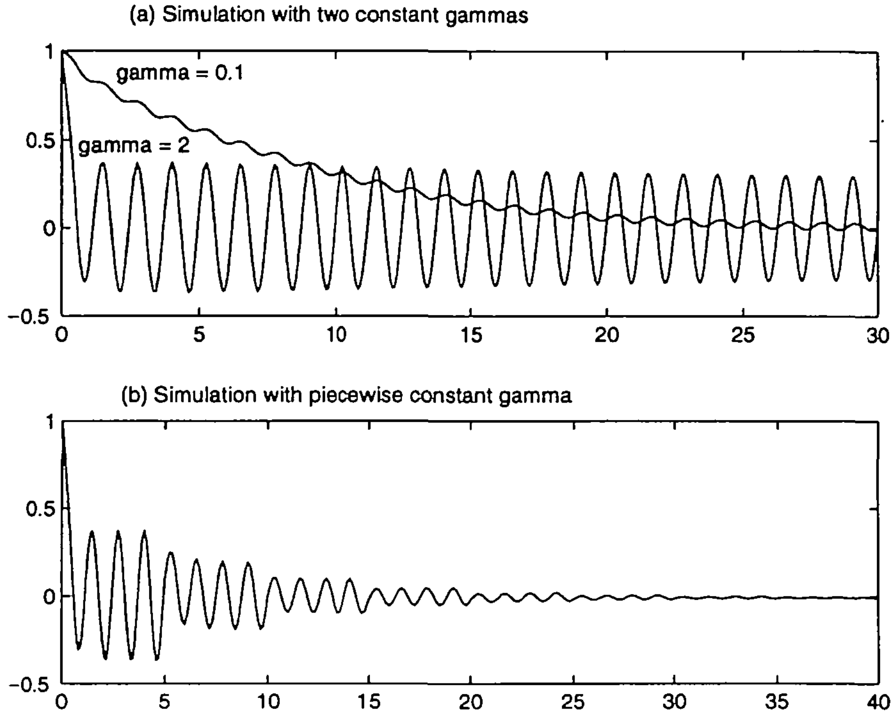

FIG. 2. Convergence in the presence of zero-mean disturbances

5. Simulation

Consider a simple first-order system

$$\dot{x} = \gamma(-2 + \sin \ln(1 + t))x(t) + \gamma \cos 5t.$$

It can be easily verified that the zero-input part has a convergence rate γ, and that the bound on the disturbance input $\gamma \cos 5t$ is also γ. The standard bounded-input–bounded-output result will give a bound of 1 on the output error, independent of γ. Applying the result of the last section, we know that the output error $x(t)$ converges to zero as $t \rightarrow 0$ and as $\gamma \rightarrow 0$. Figure 2 shows the simulation results with this system. In both part (a) and part (b), the horizontal axis is time in seconds and the vertical axis is the output error. Part (a) shows two simulation results, each with a different constant γ. It can be easily seen that the output error converges to its steady-state size much faster with a larger γ than with a smaller γ. However, the smaller γ leads to a much smaller steady-state error. To demonstrate the results of the previous section more directly, part (b) of Fig. 2 shows a simulation result clearly indicating that the output error converges to zero as $t \rightarrow 0$ and as $\gamma \rightarrow 0$. In this simulation, we have used a piecewise constant γ that is adjusted to decrease with time. Specifically, we have started with $\gamma = 2$, and reduce it by half every five seconds. Other strategies for reducing γ can also be considered.

6. Conclusion

In this paper, we have seen that, when disturbances are known to have zero ergodic mean, stronger convergence results can be achieved than in the bounded-disturbance case. It is interesting to note that the problem boils down to a converse averaging problem. The fact that exponential stability of a system does not imply the stability of its averaged form makes this an interesting problem overlooked by many researchers.

Noting that it is the fast-changing parametric excitation which stabilizes the un-averaged system we observe another interesting question: how can we introduce variations to our system coefficients to drive our system parametrically to stability? We believe that this is fundamentally related to the parameter-convergence problem in adaptive control under persistence of excitation.

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Appendix

LEMMA (Gronwall–Bellman) Let $m, \nu, h \in C[\mathbb{R}_+, \mathbb{R}_+]$ and suppose that

$$m(t) \leqslant h(t) + \int_{t_0}^{t} \nu(s) m(s)\, \mathrm{d}s \ (t \geqslant t_0).$$

Then, for $t \geqslant t_0$,

$$m(t) \leqslant h(t) + \int_{t_0}^{t} [\nu(s)h(s)] \exp\left(\int_s^t \nu(\sigma)\, \mathrm{d}\sigma\right) \mathrm{d}s.$$

If h is differentiable, then

$$m(t) \leqslant h(t_0) \exp\left(\int_{t_0}^{t} \nu(s)\, \mathrm{d}s\right) + \int_{t_0}^{t} h'(s) \exp\left(\int_s^t \nu(\sigma) \mathrm{d}\sigma\right) \mathrm{d}s.$$

Proof. See Lakshikantham *et al.* (1989: p.4). □

LEMMA S (stability of slowly varying systems: Desoer 1969) Consider a linear time-dependent system

$$\dot{x}(t) = A(t)x(t), \quad x(0) = x_0.$$

If $-\mathrm{Re}\lambda_i(A(t)) \geqslant \sigma > 0$ for all i and all t, then there is an ε^* such that, for all $\varepsilon \in (0, \varepsilon^*)$,

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} A(t) \right\| \leqslant \varepsilon\sigma$$

implies the exponential stability of 0.