Relationship Between Amplifier Settling Time and Pole-Zero Placements for Second-Order Systems*

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Abstract-**Due to the fact that a nonlinear equation has to be solved to determine the settling time of a linear time-invariant system, the relationship between the pole-zero constellation and settling time is not readily apparent. This makes it difficult to design amplifiers with optimal or near-optimal settling performance. This work attempts to instill a deeper understanding of how pole-zero placement relates to the settling performance of second-order systems.**

I. INTRODUCTION

Second-order models are frequently used to characterize the behavior of linear time-invariant systems and have been widely studied. However, due to the fact that a nonlinear equation has to be solved to determine the settling time, the relationship between the pole locations and settling time is not readily apparent.

Excepting the facts that moving the poles further into the left half-plane (LHP) results in faster settling and that the characteristic equations of a systems with widely spaced poles can often be approximated as first-order, most designers have limited insight into exactly how pole placements relate to settling time.

Designer intuition is especially lacking for cases that include a finite zero in the transfer function. In 1963, using a normalized transfer function, Waldhauer [1] demonstrated that a closely-spaced low-frequency pole and zero results in an often undesirable slow-settling component in the transient response. He derived an expression for the maximum amount of closed-loop pole-zero mismatch that can be tolerated to ensure that the unwanted settling component is no larger than a specified level. Finally, he mapped the closed-loop mismatch requirements into open-loop accuracy requirements using the standard negative feedback equation. Kamath, Meyer, and Gray [2] later extended his work to include finite amplifier slew effects.

The treatments in [1] and [2] only consider the special case of a second-order system with a closely spaced pole and zero that are positioned much lower in frequency than the other system pole. Many other types of physically realizable and advantageous second-order systems are frequently encountered but several unanswered questions must be addressed to derive optimal benefit from these systems. How fast will these alternative systems settle to a specified accuracy level? Where should the poles and zeros be placed to obtain the best possible settling performance? How sensitive to process, temperature, and aging will the resultant systems be?

To keep the following analysis simple, it is assumed throughout that the systems are linear. Nonlinearities such as finite slew capabilities of amplifiers will be neglected.

Most amplifier structures that have accurate gain requirements exploit negative feedback to stabilize the gain. Such systems are often characterized by two sets of poles, those of the open-loop amplifier, and those of the closed-loop amplifier. The pole locations cited throughout this paper refer to the *closed-loop* pole locations which are the positions the poles migrate to after the feedback loop is closed.

Various definitions of settling time exist. For this paper it is assumed that the settling time is the minimum amount of time that must elapse after a step change in the input before it can be guaranteed that the present and all future values of the output will lie within a specified tolerance of the output signal's asymptotic value. The specified tolerance is characterized by the parameter h as depicted in Fig. 1.

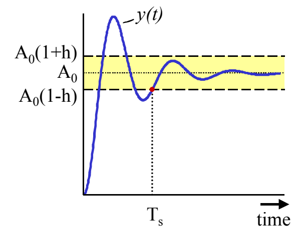

Fig. 1. Graphical depiction of settling time

All-pole systems are considered in Section II. Systems that contain a finite LHP zero are addressed in Section III. Separate subsections are provided in each case to consider systems with real and complex poles.

II. ALL-POLE SECOND-ORDER SYSTEM

The settling time of an arbitrary asymptotically-stable all-pole second-order system is considered in this section. The cases of real and complex poles are considered separately in Subsections *A* and *B* respectively.

A. Second-Order System with Real Left Half-Plane Poles

Fig. 2(a) depicts a linear system with two real left half-plane (LHP) poles. For this discussion, the lower frequency pole is denoted as P_1 while the higher frequency pole is denoted as P_2.

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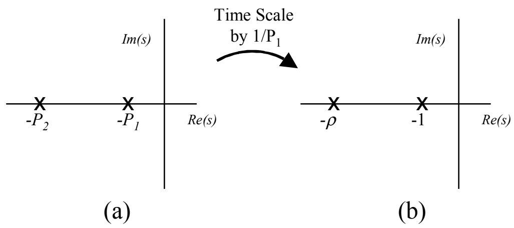

Fig. 2. Pole-zero map of a linear system with two real LHP poles (a) original system (b) after time scaling by $1/P_1$ (the *normalized system*)

The settling time, T_s, is a function of P_1, P_2, and the level of settling accuracy required, h. Thus, there are 3 independent degrees of freedom to consider. By utilizing the time-scaling property of the Laplace Transform to normalize the transfer function, the number of variables required for the visualization can be reduced from 3 to 2. The time-scaling property of the Laplace Transform is given by:

$$f(a \cdot t) \Leftrightarrow \frac{1}{a} F\left(\frac{s}{a}\right) \tag{1}$$

Scaling the time-domain response corresponds to frequency scaling its Laplace Transform. Therefore, as depicted in Fig. 2(b), time scaling by $1/P_1$ results in a translation of the system poles in the frequency domain so that the lowest frequency pole lies at s=-1 and the higher frequency pole is located at -ρ where:

$$\rho = \frac{P_2}{P_1} \tag{2}$$

The time-scaled system will be referred to as the *normalized system*. The number of degrees of freedom that affect the settling time in the normalized system is one less than that of the original system because the lowest frequency pole is always situated at s=-1.

Fig. 3 contains a plot of the normalized settling time obtained by simulation for values of ρ and h that are commonly encountered in amplifier design.

The normalized tables or equivalently, the normalized data in the plot of Fig. 3 can be used to determine the settling time of an arbitrary second-order system with real poles simply by reading the correct value out of the table and denormalizing the result. The following example illustrates the technique.

Example: Second-Order System with Real Poles

Suppose we have an application that requires settling to 0.01% accuracy and a 2nd order system with closed-loop poles located at -18MHz and -10 MHz. How much time is required for settling? Since

$$P_1 = 10\text{ MHz and } P_2 = 18\text{ MHz} \therefore P_2/P_1 = 1.8 \tag{3}$$

The point where the pole ratio line of 1.8 intersects with the 0.01% accuracy curve is approximately 10. Thus 10 is the normalized settling time. To find the actual settling time, the result must be *denormalized*:

$$T_S = 10 / P_1 = 1\ \mu\text{S} \ \blacklozenge \tag{4}$$

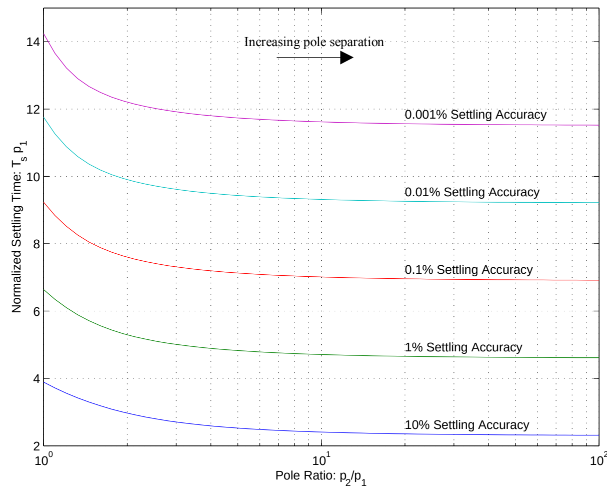

Fig. 3. Normalized settling time of a linear system consisting of two real LHP poles

Using the plot of Fig. 3, designers can gain some understanding about how settling time is affected as the pole spacing varies. Points on the y-axis (P_2/P_1 = 1.0) correspond to cases where the poles are coincident in the s-plane (critical damping). For these cases, the normalized settling time can be reasonably approximated by the following linear equation:

$$T_s = -2.54 \cdot \log_{10}(h) + 1.54 \tag{5}$$

where h is the required settling accuracy (e.g. 0.0001 for 0.01% settling).

Points further to the right in the plot represent increasing pole separation. For a fixed value of P_1, it is evident that settling time degrades as the poles are positioned closer together. It is also observed that reasonably accurate estimates of the settling time can be achieved by approximating a two pole system with a single pole when the pole-ratio exceeds a factor of approximately 10.

B. *Second-Order System with Complex LHP Poles*

Since not all closed-loop second-order systems have real poles, our study of settling time requires the consideration of systems with complex poles. A pole-zero map of a second-order system with complex LHP poles is contained in Fig. 4. The angle to the pole is denoted by θ.

The same normalization technique that was used for a system with two real LHP poles can be used for the complex case. In this case though, the normalization factor is chosen to be $\omega_p\zeta$ which ensures that the poles lie on the line Re(s) = -1. The pole-angle, θ, is not affected by scaling. Fig. 5 contains a plot of the normalized settling time as a function of θ for settling accuracies commonly encountered in amplifier design.

Points on the y-axis, (θ=0°), correspond to critical damping. Points further to the right correspond to increasingly under-damped responses. Neglecting nonlinearities such as finite amplifier slew capability, it is interesting to observe that the worst-case settling is obtained for low-Q poles. Settling to a given accuracy level actually improves as the poles initially move away from the real axis.

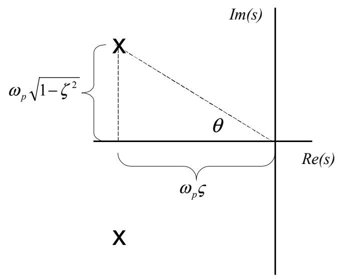

Fig. 4. Pole-zero map of a linear system with two complex LHP poles

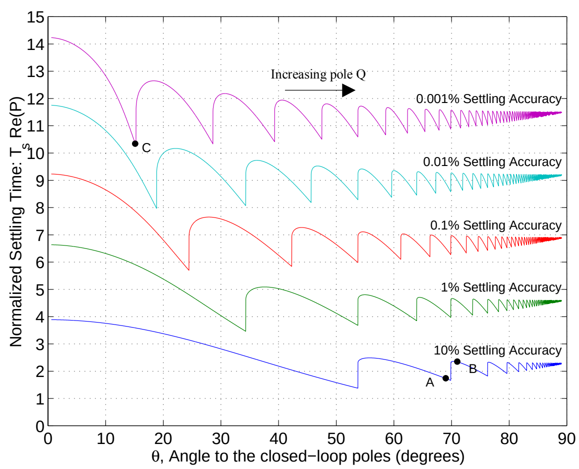

Fig. 5. Normalized settling time of a linear system consisting of two complex LHP poles

It is also interesting to observe that as θ approaches 90°, the settling time approaches a value that is equivalent to that of a system with a single pole located at s=-1.

The discontinuities, present in the curves (vertical segments in the plots) can be understood by considering points on each side of the discontinuity. For example, consider points A and B marked in the plot of Fig. 5. The corresponding time-domain step responses are shown in Fig. 6. Curve A corresponds to a θ=69° while curve B corresponds to an angle of 71°.

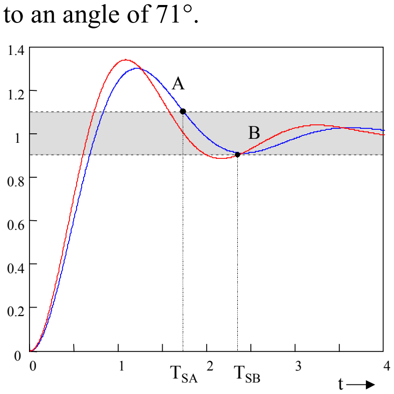

Fig. 6 Illustration used to explain the discontinuous nature of the curves in Fig. 5

In Fig. 6, the region corresponding to settling is shaded in gray. Notice that curve B crosses the lower bound of the settling region at 2.35 sec. while curve A barely remains within the settling region there. Thus curve A has a shorter settling time because it last crosses the boundary of the settling region at 1.35 sec. Although these two curves correspond to nearly equal θ's, there is a significant difference in their settling times. In fact, as θ is varied continously between A and B, at some point there is a discontinuity in the settling time as the last point in time that the step response waveform crosses the settling region boundary switches from the upper boundary to the lower boundary and vice-versa.

Using Fig. 5, designers can gain insight about the sensitivity of a system's settling time to process, aging and environmental variations. Consider a system that was intended to operate at point C marked in Fig. 5. In the presence of variations, the poles may be dislocated from their desired positions. As a consequence, the resultant angle to the poles might be slightly smaller or larger than desired. One can see from the slope of and the discontinuity present in the 0.001% settling accuracy curve near point C, that slight variations in θ result in substantial increases in settling time. It is apparent from the plot that systems with low-Q poles and those with very high-Q poles exhibit lower sensitivities to variations in the phase angle θ than systems with intermediate-Q poles.

Using Fig. 5, it is possible to determine the settling time without solving a nonlinear equation simply by reading the appropriate value off of the plot and denormalizing the result. To illustrate, consider the following example.

Example: Second-Order System with Complex Poles

As an example, assume 0.01% settling accuracy is required with closed-loop poles located at $\left(-\sqrt{3}\pm i\right)\cdot 10^6$ Hz. How much time is required for settling? The angle to the poles is given by:

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

The point where the pole angle line of 30° intersects with the 0.01% accuracy curve is approximately 9.2. Denormalizing this value results in the actual settling time.

$$T_S \approx \frac{9.2}{\left|\mathrm{Re}(P)\right|} = \frac{9.2}{\sqrt{3}\cdot 10^6} \approx 5.3\mu s \quad \blacklozenge$$

III. Second-Order Systems with a LHP Zero

In this section, the settling time for an arbitrary asymptotically stable second-order system with a LHP zero is considered. The cases of real and complex poles will be considered separately in Subsections *A* and *B* respectively.

Before proceeding, some terminology needs clarification. For many years in the integrated circuit design literature, the term *doublet* has been used to refer to a pole and zero that are spaced relatively close to each other with respect to the distance to the other system poles and zeros. This is inconsistent with the terminology used in other disciplines. In the current context, the term *doublet* is more appropriately used to denote two closely spaced poles or two closely spaced zeros but not a pole and zero that are closely spaced. In other disciplines and throughout the rest of this paper, the term *dipole* is used to denote a pole and zero that are closely spaced.

A. Systems with Real LHP Poles

The pole-zero plot of a system with two real LHP poles and a LHP zero is depicted in Fig. 7.

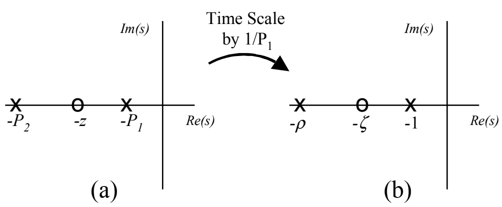

Fig. 7. Pole-zero map of a linear system with two real LHP poles and a LHP zero (a) original system (b) after time-normalization by scaling by $1/P_1$

The time-normalization technique that was used previously to simplify the visualization of the relationship between pole-zero placement and settling time is used here. In this case an additional degree of freedom is introduced due to the presence of the zero.

Designating the lower and higher frequency poles as P_1 and P_2 respectively and using a time-normalization factor of $1/P_1$ puts the normalized lowest frequency pole at s=-1. The zero and the remaining pole are designated in the normalized system as ζ and ρ respectively where $\zeta = z/P_1$ and $\rho = P_2/P_1$.

Fig. 8 contains a plot of the settling time as a function of ζ and ρ that was obtained for a settling accuracy requirement of 0.01%. In an attempt to cover the points commonly of interest to amplifier designers, ζ was varied from 1E-4 to 1E6 while ρ was swept from 1 to 1E6.

The most distinguishing feature of Fig. 8 is the sharp reduction in settling time for systems with ζ's in the vicinity of 1.0. Dramatic improvements in settling time occur in these cases due to the pole-zero cancellation that occurs as the zero passes over the low-frequency pole located at s=-1. These cases will be examined in more detail later.

The zero passes over and cancels the higher frequency pole as well. The points where this occurs do not stand out on the plot because in those cases the settling time is dominated by the low-frequency pole's slow-settling component.

Settling time diverges as ζ approaches zero. As a result, it is not advantageous to position the zero at frequencies significantly lower than that of the dominant pole. One can observe that for a fixed value of P_1, settling time improves as the pole spacing is widened.

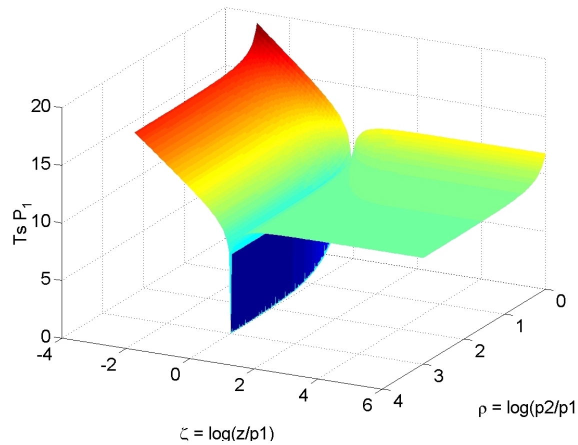

Fig. 8. Normalized settling time as a function of $\rho=P_2/P_1$ and $\zeta=z/P_1$ for a settling accuracy of 0.01%

Low-frequency dipoles are commonly avoided because they result in slow-settling components in the transient response [1][3]. Depending upon the amount of mismatch and the accuracy requirements of the application, these slow-settling components may or may not be significant. Thus, to determine how much mismatch is allowable for a given settling accuracy and speed requirement, a careful study of the sensitivity of settling time to pole-zero mismatch is warranted. In this regard, we will focus our attention on the settling behavior in a region around where $\zeta=z/P_1=1$. Observe from Fig. 8 that in the designated region, the settling time depends upon $\rho=P_2/P_1$. There, the settling time improves as the pole ratio, ρ, increases. To more clearly visualize the relationship, the case where the poles are spaced closely together is first considered. Later the case of larger pole ratios is investigated.

Fig. 9 contains a plot of the settling time for different settling accuracy requirements in the region around where the low-frequency pole cancellation occurs for cases where the poles are nearly coincident (ρ=1.001).

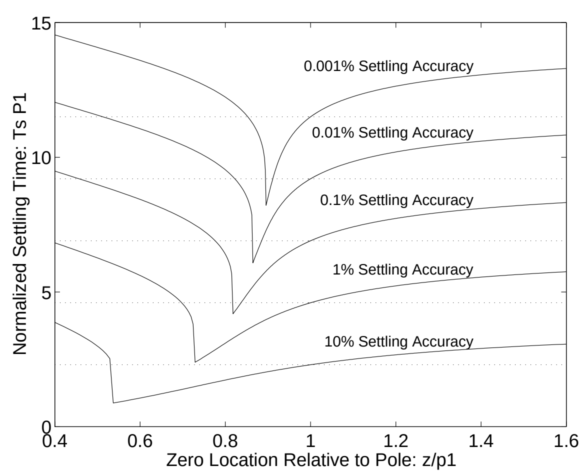

Fig. 9. Normalized settling time for closely spaced poles (p2/p1 = 1.001)

The 5 horizontal dashed lines in Fig. 9 represent the settling times that result when exact cancellation of the low-frequency poles occurs for each of the different settling accuracy cases (i.e. the topmost line corresponds to the 0.001% settling accuracy case, the next lower line to 0.01%, and so on). Contrary to statements in [1], it is observed that settling time is not minimized when exact cancellation of the low-frequency dipole occurs. Rather, the settling time is minimized when the zero is located at frequencies that are slightly lower than the dominant pole. This observation indicates that if a method for accurately placing the zero relative to the pole could be achieved, a system with a low-frequency dipole could be built that settles faster than an equivalent system without the dipole.

Fig. 9 depicts cases where the closed-loop poles are nearly coincident in the s-plane, ($\rho \approx 1$). Infinitely many other cases are possible as well. Since all cases can not be considered here, it is instructive to consider a case with more widely spaced poles. Fig. 10 contains a blow-up of the region near where the low-frequency pole cancellation occurs for the case where $\rho = P_2/P_1 = 10$.

When cancellation of the low-frequency pole occurs, the system effectively becomes a single-pole system with a settling time determined by the location of the high-frequency pole. As a result, the larger the pole-ratio becomes, the more dramatic the improvement in settling time becomes when the low-frequency pole is cancelled. This fact is observable by comparing Fig. 10 to Fig. 9. For exact low-frequency pole-zero cancellations, the settling times of the ρ=10 cases are each 1/10'th of the ρ=1 cases. Unfortunately, however, accompanying the reduction in settling time for low-frequency pole-zero cancellation due to larger pole ratios is an increased sensitivity to pole-zero mismatch.

The sensitivity of settling time to pole-zero mismatch is a function of the pole-ratio and the required settling accuracy. As previously mentioned, by positioning the zero slightly below the low-frequency pole, the settling time for a system with a low-frequency dipole can be better than that of an equivalent system without the dipole.

To gauge the sensitivity to mismatch, one can consider the maximum allowable percentage dipole mismatch that results in settling performance that is at least as good as an equivalent system without a dipole. Fig. 11 contains a plot of such a sensitivity measure where dipole mismatch was defined as:

$$\frac{|Z - P_1|}{Z} \times 100\% \tag{6}$$

or in terms of the normalized system:

$$\frac{|\zeta - 1|}{\zeta} \times 100\% \tag{7}$$

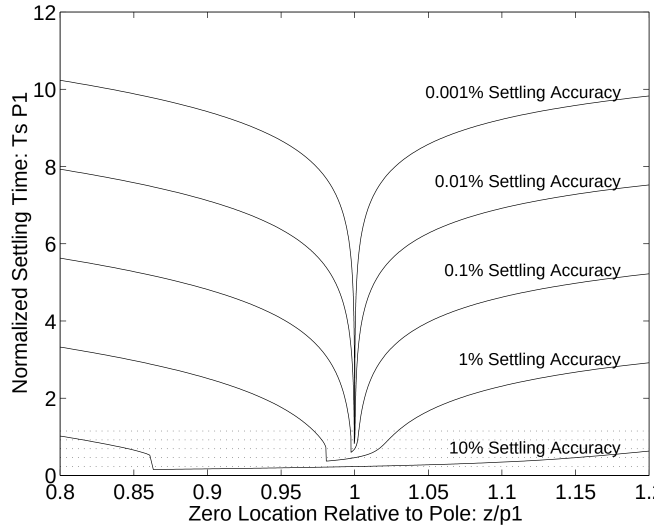

Fig. 10. Normalized settling time for widely separated poles (p2/p1 = 10)

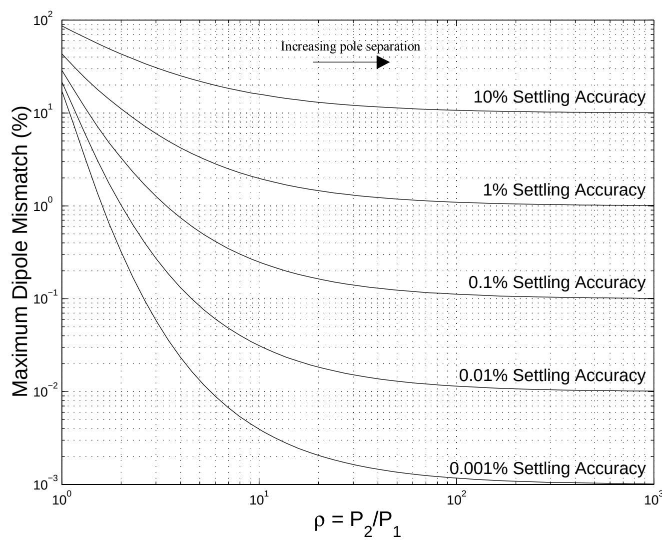

Fig. 11. Maximum allowable dipole mismatch that results in a system that settles at least as fast as one without a dipole

From Fig. 11, it is observed that to achieve settling times that are as good as an equivalent system without the low-frequency dipole, the accuracy required in the pole-zero placement increases with required settling accuracy and pole-ratio, ρ. For example, consider a system with a pole-ratio of 2 in an application that requires 0.01% settling accuracy. The point on the plot where the 0.01% curve intersects the ρ=2 line is at 1%. This indicates that to ensure the settling time of the system will be at least as fast as an equivalent system without a dipole, ζ must lie between 0.99 and 1.0. For higher settling accuracies or larger pole-ratios, the required precision in pole-zero placement increases.

B. Systems with Complex LHP Poles

A pole-zero plot of the system with two complex LHP poles located at $\omega_p\left(\zeta \pm \sqrt{1-\zeta^2}\right)$ and a LHP zero located at *s*=-*Z* is shown in Fig. 12(a).

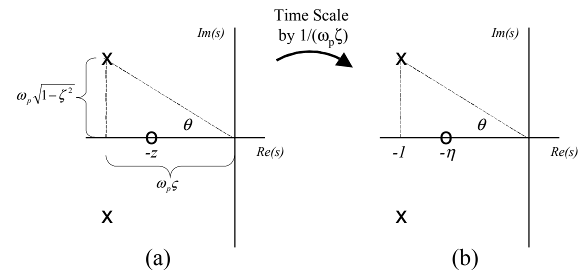

Fig. 12. Pole-zero plot of a linear system with two complex LHP poles and a LHP zero (a) original system (b) after time-normalization by scaling by $1/(\omega_p\zeta)$

Fig. 12(b) shows the effect of time normalization by $1/(\omega_p\zeta)$ on the pole and zero locations in the s-plane. After normalization, the real-parts of the poles lie at s=-1. The angle to the poles, θ, is not affected by the normalization. The post-normalization location of the zero is denoted as -η where $0 \leq \eta < \infty$.

Although the normalized settling time varies for different settling accuracy requirements, it is instructive to examine the relationship for a particular case with the knowledge that other cases will have similar characteristics. Fig. 13 contains a 2-d surface of the settling time as a function of θ and η that was obtained for settling accuracy requirement of 0.01%. In an attempt to cover the points commonly of interest to amplifier designers, θ was varied from 0.01° to 89.99° while η was swept from 0.01 to 1E6.

Assuming fixed pole locations, the effect of the location of the zero can be observed by looking at fixed-θ slices in Fig. 13. Accordingly, settling time degradation appears when the zero is located closer to the jω-axis than the poles with the initial degradation onset occurring when the zero and poles are nearly equidistant from the jω-axis and increasing as the zero moves toward the $j\omega$-axis. Conversely, it is also observed that the settling time is largely independent of the zero location when the zero is situated at distances that are at least ten times larger than the separation between the poles and the $j\omega$-axis. Therefore, in applications where settling time is important, systems with zeros closer to the $j\omega$-axis than the poles should be avoided and systems with zeros located more than ten times farther from the jω-axis than the poles can be approximated as all-pole systems.

The effect of varying pole-Q can be examined by looking at fixed-η slices of Fig. 13. From the plot it is observed that varying the angle to the poles, θ, dramatically affects settling time.

As expected, in the limit as η approaches infinity the settling response of the two-pole system with a zero asymptotically approaches the response of one without a zero. This can be observed by comparing the 0.01% curve in Fig. 5 to a fixed-η slice of Fig. 13 for a large value of η. Given that fact, inspection of Fig. 13 reveals that a two-pole system with a LHP zero can settle faster than one without a zero. That is, if the LHP zero is positioned in the vicinity of η=1, the presence of the zero can significantly reduce the settling time.

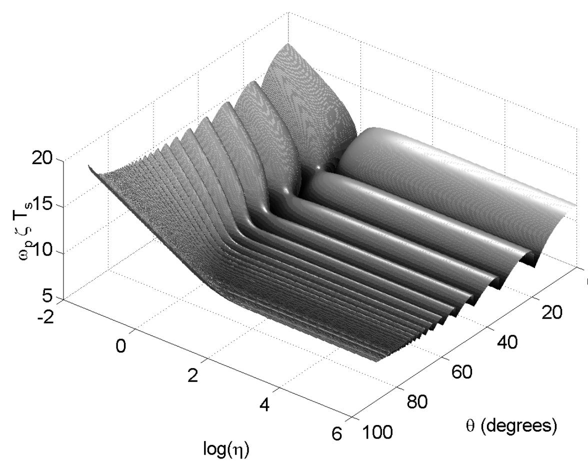

Fig. 13. Normalized settling time as a function of θ and $\eta = z/(\omega_p\zeta)$

CONCLUSION

Due to the fact that a nonlinear equation has to be solved to determine the settling time of a linear time-invariant system, the relationship between settling time and the pole and zero locations is not readily apparent. Settling characteristics of normalized second-order systems were examined to provide a comprehensive graphical relationship between the pole and zero locations and the settling time.

REFERENCES

[1] F.D. Waldhauer, "Analog Integrated Circuits of Large Bandwidth," in 1963 IEEE Conv. Rec., Part 2, pp. 200-207

[2] B.Y. Kamath, R.G. Meyer, and P.R. Gray, "Relationship Between Frequency Response and Settling Time of Operational Amplifiers," *IEEE J. Solid-State Circuits,* vol. SC-9, pp. 347-352, Dec. 1974.

[3] P.R. Gray and R.G. Meyer, "Recent Advances in Monolithic Operational Amplifier Design," *IEEE Trans. Circuits Syst.,* vol. CAS-21, pp. 317-327, May 1974.