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# Causal Inversion of Nonminimum Phase Systems [1]

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**Abstract**

Inversion of nonminimum phase systems is a challenging problem. The classical causal inverses proposed by Hirschorn result in unbounded solutions to the inverse problem where the zero dynamics are unstable. Stable inversion introduced by Chen and Paden obtains bounded but noncausal inverses for nonminimum phase systems. As a first step, this paper addresses bounded causal inversion of nonlinear nonminimum phase systems. It is shown that an optimal causal inversion problem is equivalent to a minimum energy control problem of the zero dynamics driven by a causal reference output profile. A causal inversion solution for nonlinear systems and an optimal causal inversion solution for linear systems are also proposed. Simulation results demonstrate the effectiveness of the new causal inversion approach in output tracking.

## 1 Introduction

The inverse problem is a fundamental generic problem in science and engineering. Since it widely appears in different areas, such as non-destructive evaluation [2], heat transfer [9], wave motion [6], and biomedical engineering [10], it has attracted researchers' interest for a long time. Especially in control systems areas, inversion algorithms have been applied to output tracking [5] and learning control [13].

The classic inversion approach for output tracking control uses stabilizing feedback together with feed-forward signals generated by an inverse system. The classic inverse problem was first studied by Brockett and Mesarovic [1]. Later, Silverman [11] developed a step-by-step procedure for the inversion of a class of linear multivariable systems. These linear results were extended to nonlinear real-analytic systems by Hirschorn [7] and Singh [12]. For a given desired output and a fixed initial condition, all these inversion algorithms produce causal inversion that are unbounded for nonminimum phase systems. The stable inversion approach was first developed by Chen and Paden [3] to attack a very important and difficult problem in nonlinear control: output tracking control of nonminimum phase systems. The down side is that stable inversion is noncausal.

In this paper, both causal and optimal causal inversion of nonlinear nonminimum phase systems are derived in an effort to find feed-forward signals for output tracking of a reference profile given in real time. The inversion is causal and bounded. The remainder of this paper is organized as follows. In the next section, the class of reference trajectories under consideration is defined and the causal and optimal causal inversion problems are stated. Section 3 shows that the optimal causal inversion problem is equivalent to a minimum energy control problem of the zero dynamics driven by a causal reference output profile. A causal inversion solution for nonlinear nonminimum phase systems is presented. Section 4 presented an optimal causal inversion solution and an optimal noncausal inversion solution for nonminimum phase linear systems compared with well-known stable inversion. Section 5 contains the simulation results. Finally, some concluding remarks are given in Section 6. One can refer to [14] for a more detailed introduction.

## 2 Framework and Problem Statement

First, consider a nonlinear system of the form

$$\dot{x} = f(x) + g(x)u \tag{1}$$
$$y = h(x) \tag{2}$$

defined on a neighborhood $X$ of the origin of $\Re^n$, with input $u \in \Re^m$ and output $y \in \Re^p$. $f(x)$ and $g_i(x)$ (the $i^{th}$ column of $g(x)$) for $i = 1, 2, \cdots, m$ are smooth vector fields. And $h_i(x)$ for $i = 1, 2, \cdots, p$ are smooth functions on $X$, with $f(0) = 0$ and $h(0) = 0$. For such a system, the causal inversion problem is stated as follows:

*Causal Inversion Problem:* Given a smooth reference output trajectory $y_d(t) \in L_1 \cap L_\infty$, with $y_d(t) \equiv 0$ for $t \leq 0$, find a control input $\bar{u}_d(t)$ and a state trajectory $\bar{x}_d(t)$ such that
1) $\bar{u}_d$ and $\bar{x}_d$ are bounded, and

$$\bar{u}_d(t) \to 0, \quad \bar{x}_d(t) \to 0, \quad \text{as} \quad t \to \infty$$

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2) Exact output matching is achieved

$$h(\bar{x}_d(t)) = y_d(t) \quad (3)$$

3) $\bar{x}_d(t)$ and $\bar{u}_d(t)$ are causal, that is, $\bar{x}_d(t) \equiv 0, \bar{u}_d(t) \equiv 0$ for $t \leq 0$, where $\bar{x}_d$ is the desired state trajectory, and $\bar{u}_d$ is the nominal control input.

Furthermore, by adding the following condition to the *Causal Inversion Problem*, an *Optimal Causal Inversion Problem* will be defined.
4) $\bar{u}_d$ and $\bar{x}_d$ minimize the performance index

$$J(\bar{u}_d, \bar{x}_d) = \frac{1}{2}\int_0^\infty \|\dot{\bar{x}}_d - (f(\bar{x}_d) + g(\bar{x}_d)\bar{u}_d)\|_R^2 dt$$

where $R$ is a weighting operator.

Note that, in the rest of the paper, we require $y_d(t)$ to have a compact support, that is, there exist $t_f$ such that $y_d(t) \equiv 0$ for all $t \geq t_f$, where finite for $t_f > 0$. This assumption covers a large family of practical reference trajectories. Furthermore, the development in this paper can be extended with little effort to cover signals whose certain derivatives have a compact support.

Condition (1) guarantees the stability of the internal state and input. Note that there are infinitely many $\bar{x}_d$ satisfying Equation (3). For the existence and uniqueness of the inversion solution, one can refer to [5]. Condition (3) states the causality of the solution. Condition (4) is the performance index that the optimal causal inversion has to be satisfied. Since the nominal input is used as a feed-forward signal and the state error $(x - \bar{x}_d)$ is used as a feedback signal to a stabilizing tracking controller, one must design a proper feedback controller to guarantee the asymptotic stability of the tracking error system. If $J$ is large, one may not be able to design such a feedback controller to have $e(t) \to 0$ as $t \to \infty$. So this performance index has a reasonable physical meaning. Other choices for the performance index are possible as long as they have a specific physical meaning.

In the stable inversion approach, the whole desired output trajectory is required. Compared with stable inversion, the inversions defined above are causal and bounded. The reference output profile can be given in real time.

## 3 Causal Inversion for Nonlinear Systems

Consider a nonlinear system of the form (1) and (2) with the same number of $m$ inputs and outputs and

$$\begin{aligned} y &= (y_1, y_2, \cdots, y_m)^T \\ u &= (u_1, u_2, \cdots, u_m)^T \\ h(x) &= [h_1(x), h_2(x), \cdots, h_m(x)]^T \\ g(x) &= [g_1(x), g_2(x), \cdots, g_m(x)] \end{aligned}$$

Assume that the system has a well-defined relative degree $r = \{r_1, r_2, \cdots, r_m\}^T \in N^m$ at the equilibrium point 0, that is, in an open neighborhood of 0,
(i) for all $1 \leq j \leq m$, for all $1 \leq i \leq m$, for all $k < r_i - 1$ and $k \geq 0$, and for all $x$, that is,

$$L_{g_j} L_f^k h_i(x) = 0 \quad (4)$$

(ii) the $m \times m$ matrix $\beta(x) \triangleq L_g^{(1)} L_f^{(r-1)} h(x)$ is nonsingluar. Note that since the control $u$ does not appear explicitly in Equation (2), we have $r_i \geq 1$ for all $i$. Therefore, $r - 1 \in N^m$ and the operation in the definition of $\beta$ is well defined.

Under this assumption, the system can be partially linearized. To do this, we differentiate $y_i$ until at least one $u_i$ appears explicitly. This will happen at exactly the $r_i^{th}$ derivative of $y_i$ due to (4). Define $\xi_k^i = y_i^{(k-1)}$ for $i = 1, \cdots, m$ and $k = 1, \cdots, r_i$, and denote [4]

$$\begin{aligned} \xi &= (\xi_1^1, \xi_2^1, \cdots, \xi_{r_1}^1, \xi_1^2, \cdots, \xi_{r_2}^2, \cdots, \xi_{r_m}^m)^T \\ &= (y_1, \dot{y}_1, \cdots, y_1^{(r_1-1)}, y_2, \cdots, y_2^{(r_2-1)}, \cdots, y_m^{(r_m-1)})^T \end{aligned}$$

Choose $\eta$, an $n - |r|$ dimensional function on $\Re^n$, such that $(\xi^T, \eta^T)^T = \psi(x)$ forms a change of coordinate with $\psi(0) = 0$ [8]. In this new coordinate system, the system dynamics of Equations (1) and (2) become

$$\begin{cases} \dot{\xi}_1^i = \xi_2^i \\ \vdots \\ \dot{\xi}_{r_i-1}^i = \xi_{r_i}^i \qquad \text{for } \; i = 1, \cdots, m \\ \dot{\xi}_{r_i}^i = \alpha_i(\xi, \eta) + \beta_i(\xi, \eta) u \end{cases}$$

$$\dot{\eta} = q_1(\xi, \eta) + q_2(\xi, \eta) u$$

which, in a more compact form, is equivalent to

$$y^{(r)} = \alpha(\xi, \eta) + \beta(\xi, \eta) u \quad (5)$$

$$\dot{\eta} = q_1(\xi, \eta) + q_2(\xi, \eta) u \quad (6)$$

where

$$\begin{aligned} \alpha(\xi, \eta) &= L_f^r h(\psi^{-1}(\xi, \eta)) \\ \beta(\xi, \eta) &= L_g^1 L_f^{r-1} h(\psi^{-1}(\xi, \eta)) \end{aligned}$$

$\alpha(0,0) = 0$ since $f(0) = 0$, and $\alpha_i$ and $\beta_i$ are the $i^{th}$ row of $\alpha$ and $\beta$ respectively. By the relative degree assumption, $\beta(\xi, \eta)$ is nonsingular, the following feedback control law

$$u \triangleq \beta^{-1}(\xi, \eta)[v - \alpha(\xi, \eta)] \quad (7)$$

is well defined and partially linearizes the input output dynamics relationship into a chain of integrators, $y^{(r)} = v$, where $v \in \Re^m$ is the new control input. For the inversion problem, we require $y(t) \equiv y_d(t)$ which leads to:

$$v = y_d^{(r)}$$

$$\xi = \xi_d \triangleq (y_{d_1}, \dot{y}_{d_1}, \cdots, y_{d_1}^{(r_1-1)}, y_{d_2}, \cdots, y_{d_2}^{(r_2-1)}, \cdots, y_{d_m}^{r_m-1})^T \quad (8)$$

Equation (6) becomes the zero dynamics driven by the reference output trajectory,

$$\dot{\eta} = p(y_d^{(r)}, \xi_d, \eta) \tag{9}$$

where

$$p(y_d^{(r)}, \xi_d, \eta) = q_1(\xi_d, \eta) + q_2(\xi_d, \eta)\beta^{-1}(\psi^{-1}(\xi_d, \eta)) [y_d^{(r)} - \alpha(\psi^{-1}(\xi_d, \eta))].$$

For reference trajectories with compact support, the reference dynamics become autonomous zero dynamics for $t$ outside the compact interval $[t_0, t_f]$. Assume $\eta = 0$ is a hyperbolic equilibrium point of the autonomous zero dynamics. Linearizing the right hand side of Equation (9) at the equilibrium point $\eta = 0$ gives

$$\dot{\eta} = A\eta + b(t) \tag{10}$$

where

$$A = \frac{\partial p}{\partial \eta}(y_d^{(r)}, \xi_d, \eta)|_{\eta=0,\ \xi_d=0,\ y_d^{(r)}=0}$$
$$b(t) = p(y_d^{(r)}, \xi_d, \eta) - A\eta$$

For a real matrix $A$, there exist an invertible $(n-r) \times (n-r)$ matrix $P_1$, such that $\bar{J} = P_1^{-1}AP_1$, where $\bar{J}$ is the real Jordan form of $A$. Therefore, with the coordinate transformation $\eta = P_1[\eta_s \ \ \eta_u]^T$, the reference dynamics in the new coordinate is in real Jordan form. As a result, Equation (9) can be rewritten as:

$$\dot{\eta}_s = A_s\eta_s + B_s y_d^{(r)} + d_s(y_d^{(r)}, \xi_d, \eta_s, \eta_u) \tag{11}$$

$$\dot{\eta}_u = A_u\eta_u + B_u y_d^{(r)} + d_u(y_d^{(r)}, \xi_d, \eta_s, \eta_u) \tag{12}$$

where $A_s$ has all eigenvalues in the open left-half plane with dimension $n_s$, $A_u$ has all eigenvalues in the open right-half plane with dimension $n_u$, and $d_s(\cdot)$ and $d_u(\cdot)$ denote the higher-order-terms (H.O.T.) of the expression.

From (11) and (12), two dynamic equations are defined as follows:

$$\dot{\bar{\eta}}_s = A_s\bar{\eta}_s + B_s y_d^{(r)} + d_s(y_d^{(r)}, \bar{\xi}_d, \bar{\eta}_s, \bar{\eta}_u), \bar{\eta}_s(0) = 0 \tag{13}$$

$$\dot{\bar{\eta}}_u = A_u\bar{\eta}_u + B_u y_d^{(r)} + d_u(y_d^{(r)}, \bar{\xi}_d, \bar{\eta}_s, \bar{\eta}_u) + \bar{v}, \bar{\eta}_u(0) = 0 \tag{14}$$

where $\bar{\xi}_d = \xi_d$ and $\bar{v}$ is to be chosen to reach the asymptotic stability of (13) and (14). By selecting $\bar{v} = K\bar{\eta}_u - d_u(y_d^{(r)}, \bar{\xi}_d, \bar{\eta}_s, \bar{\eta}_u)$, (14) becomes

$$\dot{\bar{\eta}}_u = (A_u + K)\bar{\eta}_u + B_u y_d^{(r)}, \ \ \bar{\eta}_u(0) = 0 \tag{15}$$

with $K$ chosen such that $(A_u + K)$ is Hurwitz. Then solving (15), we get a bounded solution $\bar{\eta}_u(t)$ and $\bar{\eta}_u(t) \to 0$ as $t \to \infty$. When restricting attention to a sufficiently small neighborhood of the equilibrium point, $d_s(\cdot)$, which is the H.O.T. in (13), is dominated by linear terms. Notice that $A_s$ is Hurwitz. By plugging $\bar{\eta}_u$ into (13) and regarding it as an external input, (13) lends to a bounded solution $\bar{\eta}_s(t)$ since $A_s$ is Hurwitz. We assume $y_d^{(i)} \to 0$ for $i = 1, \cdots, r$ as $t \to \infty$. Furthermore, $\bar{\eta}_s(t) \to 0$ as $t \to \infty$ is obtained. Also, $\bar{\xi}_d = 0$ for $t \leq 0$ and for $t \geq t_f$.

Since the system has a well-defined relative degree at the equilibrium point 0, $\psi(x) = [\xi^T \ \ \eta^T]^T = [\xi^T, \ \ [\eta_s \ \ \eta_u]P_1^T]^T$ defines a local diffeomorphism. Its inverse is $x = \phi(\xi, \eta)$. Define $\bar{\eta} = P_1[\bar{\eta}_s \ \ \bar{\eta}_u]^T$. Let

$$\bar{x}_d = \phi(\bar{\xi}_d, \bar{\eta}) \tag{16}$$

$$\bar{u}_d = \beta^{-1}(\bar{\xi}_d, \bar{\eta})[y_d^{(r)} - \alpha(\bar{\xi}_d, \bar{\eta})] \tag{17}$$

Then $\bar{x}_d$ and $\bar{u}_d$ are bounded, and $\bar{x}_d(t), \bar{u}_d(t) \to 0$ as $t \to \infty$. And by the definition of $\bar{\xi}, h(\bar{x}_d) = y_d$ and $L_f^i h(\bar{x}_d) = y_d^{(i)}$ for $i \leq r$ are obtained.

Thus a casual inversion solution to nonlinear systems has been provided. The algorithm can be summarized in the following theorem.

**Theorem 1** : *Given a smooth reference output trajectory $y_d(t)$ with compact support, consider the system described by (1) and (2), with $p = m$, and where this system has a well defined relative degree and its zero dynamics have a hyperbolic equilibrium at 0. Then a causal inversion is given by (16) and (17), where $\bar{\xi}_d, \bar{\eta}_s$, and $\bar{\eta}_u$ are solved by (8), (13) and (14) respectively, and $\bar{v}$ is given by $\bar{v} = K\bar{\eta}_u - d_u(y_d^{(r)}, \bar{\xi}_d, \bar{\eta}_s, \bar{\eta}_u)$ with $K$ chosen such that $(A_u + K)$ is Hurwitz.*

Furthermore, define $J_\phi = [\frac{\partial\phi}{\partial\bar{\xi}_d} \ \ \frac{\partial\phi}{\partial\bar{\eta}}]$, and $\tilde{P} = \begin{bmatrix} I_{(n-n_1)\times(n-n_1)} & O_{(n-n_1)\times n_1} \\ O_{n_1\times(n-n_1)} & P_1 \end{bmatrix}$, where $n_1 = n_s + n_u$, then it follows

$$\begin{aligned}
&\dot{\bar{x}}_d - (f(\bar{x}_d) + g(\bar{x}_d)\bar{u}_d) \\
&= J_\phi \begin{bmatrix} \dot{\bar{\xi}}_d \\ \dot{\bar{\eta}} \end{bmatrix} - (f(\bar{x}_d) + g(\bar{x}_d)\bar{u}_d) \\
&= J_\phi\tilde{P}\left( \begin{bmatrix} \dot{\bar{\xi}}_d \\ \dot{\bar{\eta}}_s \\ \dot{\bar{\eta}}_u \end{bmatrix} - \begin{bmatrix} \dot{\bar{\xi}}_d \\ A_s\bar{\eta}_s + B_s y_d^{(r)} + d_s(y_d^{(r)}, \bar{\xi}_d, \bar{\eta}_s, \bar{\eta}_u) \\ A_u\bar{\eta}_u + B_u y_d^{(r)} + d_u(y_d^{(r)}, \bar{\xi}_d, \bar{\eta}_s, \bar{\eta}_u) \end{bmatrix} \right) \\
&= J_\phi\tilde{P} \begin{bmatrix} 0 \\ 0 \\ \bar{v} \end{bmatrix}
\end{aligned}$$

Setting $P = J_\phi\tilde{P}$ and choosing $R = [P^{-1}(\cdot)]^T P^{-1}(\cdot)$, it yields

$$\|\dot{\bar{x}}_d - (f(\bar{x}_d) + g(\bar{x}_d)\bar{u}_d)\|_R^2 = \|\bar{v}\|_2^2 \tag{18}$$

**Definition 1** : *Minimum Energy Control Problem (MECP)*

$$J(\bar{v}^*) = \min_{\bar{v}} \frac{1}{2}\int_0^\infty \|\bar{v}\|_2^2 dt \tag{19}$$

subject to

$$\bar{\xi}_d = \xi_d \tag{20}$$

$$\dot{\bar{\eta}}_s = A_s\bar{\eta}_s + B_s y_d^{(r)} + d_s(y_d^{(r)}, \bar{\xi}_d, \bar{\eta}_s, \bar{\eta}_u), \;\; \bar{\eta}_s(0) = 0 \quad (21)$$

$$\dot{\bar{\eta}}_u = A_u\bar{\eta}_u + B_u y_d^{(r)} + d_u(y_d^{(r)}, \bar{\xi}_d, \bar{\eta}_s, \bar{\eta}_u) + \bar{v}, \;\; \bar{\eta}_u(0) = 0 \quad (22)$$

Since each step above is equivalent, these results can be summarized as follows:

**Theorem 2** : *Let the system described by (1) and (2), with $p = m$, has a well defined relative degree and its zero dynamics have a hyperbolic equilibrium at 0. Then the optimal causal inversion problem has a solution that is provided by the solution of the Minimum Energy Control Problem.*

Since $n_u < n$, MECP reduces the order of the system. Thus solving MECP is easier than solving the original problem. If $n_u$ is small enough, for an extreme case, $n_u = 1$, MECP becomes a simple scalar optimal control problem.

For nonlinear systems, the stable and unstable subspaces are coupled. If the properties of $d_s$ and $d_u$ are known, finding the minimum $\bar{v}$ stabilizing the coupled systems becomes possible. If $d_s$ and $d_u$ are totally unknown, one solution suggests choosing $\bar{v}$ to cancel $d_u$ and adding a minimum state feedback control to stabilize (22), then solve for $\bar{\eta}_u$. Afterwards, substituting this into (21) as feedback control to solve for $\bar{\eta}_s$ as was done when solving the causal inversion problem. For linear systems, the separation is easily done through eigenspace decomposition. Thus the stable and unstable subspace are decoupled. The procedure can be seen from the following section.

## 4 Optimal Causal Inversion Solution for Linear Systems

### 4.1 Optimal Causal Inversion Solution

Consider a linear system of the form

$$\dot{x} = Ax + Bu \quad (23)$$

$$y = Cx \quad (24)$$

where $x \in \Re^n$, input $u \in \Re^m$, output $y \in \Re^m, A \in \Re^{n\times n}, B \in \Re^{n\times m}$, and $C \in \Re^{m\times n}$, with a well-defined vector relative degree.

Given a smooth reference output trajectory $y_d(t)$ with $y_d(t) \equiv 0$ for $t \le 0$ and $t \ge t_f$, consider finding a control input $\bar{u}_d(t)$ and a state trajectory $\bar{x}_d(t)$ such that

1) $\bar{u}_d$ and $\bar{x}_d$ are bounded, and

$$\bar{u}_d(t) \to 0, \;\; \bar{x}_d(t) \to 0, \;\; as \;\; t \to \infty$$

2) Exact output tracking is achieved:

$$C\bar{x}_d(t) = y_d(t)$$

3) $\bar{x}_d(t)$ and $\bar{u}_d(t)$ are causal, that is, $\bar{x}_d(t) \equiv 0, \bar{u}_d(t) \equiv 0$ for $t \le 0$

4) $\bar{u}_d$ and $\bar{x}_d$ minimize the performance index

$$J(\bar{u}_d, \bar{x}_d) = \tfrac{1}{2}\int_0^\infty \|\dot{\bar{x}}_d - (A\bar{x}_d + B\bar{u}_d)\|_R^2 dt$$

where $R$ is a symmetric positive definite $n \times n$ matrix.

For the inversion problem, let $y \equiv y_d$ and $u = u_d$ in (23) and (24). Then the system becomes

$$\dot{x} = Ax + Bu_d \quad (25)$$

$$y_d = Cx \quad (26)$$

Differentiating $y_d(t)$ until $u_d$ appears explicitly in the right hand side, solving for $u_d$, and substituting into (25) and (26) yields

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}y_d^{(r)}(t) \quad (27)$$

$$u_d(t) = \bar{C}x(t) + \bar{D}y_d^{(r)}(t) \quad (28)$$

where $\bar{A} \in \Re^{n\times n}, \bar{B} \in \Re^{n\times m}, \bar{C} \in \Re^{m\times n}, \bar{D} \in \Re$ and

$$\bar{A} = A - B(CA^{(r-1)}B)^{-1}CA^r$$
$$\bar{B} = B(CA^{(r-1)}B)^{-1}$$
$$\bar{C} = -(CA^{(r-1)}B)^{-1}CA^r$$
$$\bar{D} = (CA^{(r-1)}B)^{-1}$$

Performing a change of variables so that

$$x_d = Pz = P[\xi, \eta_s, \eta_u]^T, \quad (29)$$

it leads to

$$\dot{\xi} = A_c\xi + B_c y_d^{(r)} \quad (30)$$

$$\dot{\eta}_s = A_s\eta_s + B_s y_d^{(r)} \quad (31)$$

$$\dot{\eta}_u = A_u\eta_u + B_u y_d^{(r)} \quad (32)$$

$$u_d = [C_c \;\; C_s \;\; C_u][\xi \;\; \eta_s \;\; \eta_u]^T + \bar{D}y_d^{(r)} \quad (33)$$

where $A_c, A_s, A_u$ are real Jordan matrices of suitable dimensions; $A_c$ has $r$ eigenvalues at zero; $A_s$ has all eigenvalues in the open left-half plane; $A_u$ has all eigenvalues in the open right-half plane. Thus the inverse system has been decoupled to center, stable, and unstable subsystems.

Picking the transformation matrix $P$ so that the center subsystem is a simple chain of $r$ integrators and solving for $\xi$ gives

$$\xi = [y_d, \dot{y}_d, \cdots, y_d^{(r-1)}]^T \quad (34)$$

From (31) and (32), we form

$$\dot{\eta}_s = A_s\eta_s + B_s y_d^{(r)}, \;\; \eta_s(0) = 0 \quad (35)$$

$$\dot{\bar{\eta}}_u = A_u\bar{\eta}_u + B_u y_d^{(r)} + \bar{v}, \;\; \bar{\eta}_u(0) = 0$$

Set $\bar{\xi}_d = \xi$ and $\bar{\eta}_s = \eta_s$. Let

$$\bar{x}_d = P[\bar{\xi}_d, \bar{\eta}_s, \bar{\eta}_u]^T \quad (36)$$

$$\bar{u}_d = C_c\bar{\xi}_d + C_s\bar{\eta}_s + C_u\bar{\eta}_u + \bar{D}y_d^{(r)}. \quad (37)$$

Choosing $R = [P^{-1}]^T P^{-1}$, then by Theorem 2 in Section 3, the optimal causal inversion solution of linear systems (23) and (24) is provided by the solution of the following *Minimum Energy Control Problem* (MECP)

$$J(\bar{v}^*) = \min_{\bar{v}} \frac{1}{2} \int_0^\infty \|\bar{v}\|_2^2 dt \qquad (38)$$

subject to

$$\dot{\bar{\eta}}_u = A_u \bar{\eta}_u + B_u y_d^{(r)} + \bar{v}, \quad \bar{\eta}_u(0) = 0 \qquad (39)$$

To derive the main result, consider the following lemma.

**Lemma 1** : *Consider a linear system with continuous disturbance described by*

$$\dot{x} = Ax + u + D(t), \;\; x(0) = 0 \qquad (40)$$

*where u, x are finite dimensional vectors depending on the time t and D(t) is a piecewise differentiable function. Then the minimum energy control defined by* $J(u^*) = \min_u \frac{1}{2} \int_0^\infty \|u\|_2^2 dt$, *is given by* $u^* = K(t)x + M(t)$, *where K(t) is the solution of the Riccati Equation*

$$KA + A^T K + K^2 = 0 \qquad (41)$$

*and*

$$M(t) = -K \int_\infty^t exp[A(t-\tau)]D(\tau)d\tau \qquad (42)$$

*Proof:* See [14].

**Assumptions A1** : $y_d^{(r)}(t)$ is a piecewise differentiable function.

Let $D(t) = B_u y_d^{(r)}(t)$. Using Lemma 1, the minimum energy control must be $\bar{v}^* = K(t)x + M(t)$, where $K(t)$ and $M(t)$ satisfy (41) and (42) respectively. In order to obtain $M(t)$, the future of the D(t) must be known. For causal inversion, there is no any future information, so we assume $D(t) = 0$. Thus the optimal solution of (38) and (39) is given by $\bar{v}^* = K\bar{\eta}_u$. Furthermore, from (41), it yields $\sigma(A_u + K) = \sigma(-K^{-1}A_u^T K)$, where $\sigma(A_u)$ is the spectrum of $A_u$. Finally, this leads to $\sigma(A_u + K) = -\sigma(A_u)$.

Since $A_u$ has all the eigenvalues in the open right-half plane, the optimal control solution is chosen such that the closed-loop eigenvalues approach the reflections through the imaginary axis of the open-loop eigenvalues.

From the above argument, the conditions and an algorithm to obtain an optimal causal inversion solution for linear systems can be summarized in the following theorem.

**Theorem 3** : *Consider a linear system described by (23) and (24) under Assumption A1; then*

*1) to solve the optimal causal inversion problem is equivalent to solving the minimum energy control problem (38) and (39);*
*2) the solution of (38) and (39) is given by* $\bar{v}^* = K\bar{\eta}_u$, *where K is chosen such that* $\sigma(A_u + K) = -\sigma(A_u)$ *is satisfied;*
*3)the optimal causal inversion solution for the system is given by (36) and (37), where* $\bar{\xi}_d, \bar{\eta}_s$, *and* $\bar{\eta}_u$ *are solved by (34), (35), and (39) respectively.*

### 4.2 Optimal Noncausal Inversion Solution

In fact, if given the whole trajectory of $y_d^{(r)}(t)$, an optimal noncausal solution can be found using the above approach.

Assume $D(t) = B_u y_d^{(r)}(t)$ satisfies Assumption A1. From (42), it follows that $M = -K\eta_u$. Then the minimum energy control becomes $\bar{v}^* = K\bar{\eta}_u - K\eta_u$. Thus $\bar{\eta}_u$ satisfies $\dot{\bar{\eta}}_u = (A_u + K)\bar{\eta}_u + B_u y_d^{(r)} - K\eta_u$. Combined with (32), it follows that $\dot{\tilde{\eta}}_u = (A_u + K)\tilde{\eta}_u, \tilde{\eta}_u(0) = -\eta_u(0)$, where $\tilde{\eta}_u = \bar{\eta}_u - \eta_u$. Since $(A_u + K)$ is Hurwitz, $\bar{\eta}_u(t) \to \eta_u(t)$ as $t \to \infty$.

The exact output matching is also achieved. Compared with the stable inversion solution, the optimal noncausal solution has a zero initial condition. Comparison with the simulation results is shown in the following section.

## 5 Simulation Illustrations

In this section, optimal causal inversion is applied to a simple linear nonminimum phase system for output tracking.

Now consider a single-input single-output linear nonminimum phase system described by the following equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = -21x_1 - 4x_2 + x_3$$

This system has zeros at 7 and -3 and the causal reference output trajectory is given by:

$$y_d = \begin{cases} 0.02 - 0.02\cos(4\pi t), & t \in [0, 0.5] \\ 0, & \text{otherwise} \end{cases}$$

as shown by the solid curve in Figure 1. Then from Section 4, the following equations obtained:

$$\bar{\xi}_d = -0.25y_d$$
$$\dot{\bar{\eta}}_s = -3\bar{\eta}_s + -0.1429\dot{y}_d, \;\; \bar{\eta}_s(0) = 0$$
$$\dot{\bar{\eta}}_u = 7\bar{\eta}_u + 0.3333\dot{y}_d + \bar{v}, \;\; \bar{\eta}_u(0) = 0$$

where $\bar{v} = -14\bar{\eta}_u$, yielding $\bar{\xi}_d, \bar{\eta}_s$, and $\bar{\eta}_u$. Thus the causal inversion solution can be obtained by (36) and

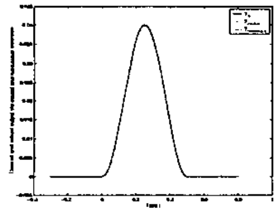

**Figure 1:** Desired and actual output trajectory via optimal causal inversion and optimal noncausal inversion approach

(37). The simulation results are shown in Figure 1. Notice that the exact output tracking is achieved for both optimal casual and noncausal approaches. Figure 2

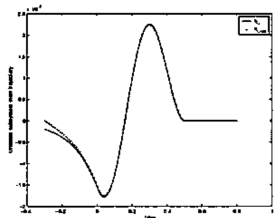

**Figure 2:** Unstable subsystem state trajectory via stable inversion and optimal noncausal inversion approach

demonstrates that the optimal noncausal solution for an unstable subsystem matches the stable inversion solution quite well. Note that the optimal noncausal solution has a zero initial condition; whereas, the stable inversion solution has a nonzero initial condition.

The above results demonstrate that the proposed casual inversion algorithm is very effective in reproducing the desired trajectories. The difference between the optimal noncausal solution and the stable inversion is shown above as well.

## 6 Conclusions

This paper has introduced the notion of causal and optimal causal inversion of nonlinear nonminimum phase systems. The optimal causal inversion problem is shown to be equivalent to a minimum energy control problem of the zero dynamics driven by reference output profile. A causal inversion solution and an optimal causal inversion solution are proposed for nonmimimum phase nonlinear and linear systems respectively. An optimal noncausal inversion solution is presented and compared with stable inversion solution. These inversion techniques are fundamental to nonlinear tracking controllers that use feed-forward signals produced by inversion in conjunction with stabilizing feedback signals. Simulation results demonstrate that the causal inversion is very effective in obtaining exact output tracking. Future work will study on efficient algorithms for constructing optimal causal inversion of nonlinear nonminimum phase systems and their applications.

## References


[1] R. W. Brockett and M. D. Mesarovic, "The reproducibility of multivariable systems," *J. Math. Anal. Appl.*, vol. 11, pp. 548-563, 1965.

[2] O. M. Bucci, L. Crocci, T. Isernia, and V. Pascazio, "An adaptive wavelet-based approach for nondestructive evaluation applications," *IEEE Antennas and Propagation Society International Symposium*, vol. 3, pp. 1756-1759, 2000.

[3] D. Chen and B. Paden, "Stable inversion of nonlinear non-minimum phase systems," *Proc. of Japan/USA Symposium on Flexible Automation*, pp. 791-797, 1992.

[4] D. Chen, "An iterative solution to stable inversion of nonlinear non-minimum phase systems," *American Control Conference*, pp. 2960-2964, 1993.

[5] S. Devasia, D. Chen, and B. Paden, "Nonlinear inversion-based output tracking," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 930-942, 1996.

[6] T. M. Habashy and R. Mittra, "On some inverse methods in electromagnetics," *J. Electromagn. Waves. Applicat.*, vol. 1, pp. 25-58, 1987.

[7] R. M. Hirschorn, "Invertibility of nonlinear control systems," *SIAM J. Contr. Optim.*, vol. 17, no. 2, pp. 289-297, 1979.

[8] A. Isidori, *Nonlinear control systems: an introduction*, Springer-Verlag, New Jersey, 1989.

[9] S. Lin and H. Chu, "Thermal uniformity of 12-in silicon wafer during rapid thermal processing by inverse heat transfer methods," *IEEE Trans. on Semiconductor Manufacturing*, vol. 13, pp. 448-456, 2000.

[10] J. C. , R. M. Leahy, and P. S. Lewis, "EEG and MEG: forward solutions for inverse methods," *IEEE Trans. on Biomedical Engineering*, vol. 46, pp. 245-259,1999.

[11] L. M. Silverman, "Inversion of multivariable linear systems," *IEEE Trans. Automat. Contr.*, vol. 14, pp. 270-276, 1969.

[12] S. N. Singh, "A modified algorithm for invertibility in nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 595-599, 1981.

[13] X. Z. Wang and D. J. Chen, "Adaptive learning control for nonminimum phase systems," *2000 IEEE International Conference on Systems, Man, and Cybernetics*, vol. 1, pp. 26-31, 2000.

[14] X. Z. Wang and D. J. Chen, "Internal report," *Iowa State University*, 2001.