Equivalent Gain Analysis for Nonlinear Operational Amplifiers*

Chengming He, Le Jin, Hanjun Jiang, Degang Chen and Randall Geiger
Department of Electrical and Computer Engineering
Iowa State University, Ames, IA, 50011, USA

Abstract

For low-voltage, low-power circuit design, the linear model for operational amplifiers is insufficient to accurately predict the static and dynamic behavior of the circuit. This paper introduces a nonlinear model for operational amplifiers based on amplifiers' DC transfer characteristic. By using this nonlinear model, the behavior of an operational amplifier with feedback network can be described more accurately. Equivalent open-loop gains of the operational amplifier for several common closed-loop applications are studied. These equivalent gains can help identify the nonlinearities in the close-loop amplifier and determine the performance of the circuit.

Index Terms – nonlinear amplifier, equivalent gain

1. Introduction

With the rapid development of semiconductor technology, the feature size of devices is reducing continuously and the threshold voltages also, accordingly. Because of the device limitation and the consideration of power dissipation, low voltage supplies become essential for integrated circuit design in deep submicron processes. However, for low voltage circuit design, the classic equations such as the square-law equation cannot precisely predict the characteristics of devices. Performance of transistors is affected by the short channel effects and becomes highly nonlinear. Under low supply voltage, the need for high gain and large output swing forces designers to exploit the performance of transistors and design circuits that work in the strongly nonlinear region. Although engineers make great effort to improve the linearity of amplifiers, they cannot eliminate the non-linearity due to the intrinsic characteristics of semiconductor devices. Actual open-loop amplifiers always have non-linearity in the transfer characteristics. Though the open-loop operational amplifiers are highly non-linear, the close-loop configuration can still give linear input-output characteristics if the feedback network is linear and the open-loop gain is sufficiently high. [1][2][3]

For a linear amplifier, the open-loop gain is well defined as $A=V_o/V_{in}$. Typically the gain is defined at the quiescent point and is extended to the whole 'linear-region' in which $A=V_o/V_{in}$ is a constant. This gain is a key parameter for various closed-loop applications of the Op Amp. However, for nonlinear amplifiers, $A=V_o/V_{in}$ may be strongly dependent on the output level and can be significant smaller than its value at the quiescent point. It is unclear what "gain" values of open-loop amplifiers should be used to predict the behavior of the close-loop amplifiers. It thus is necessary to look into various typical applications of nonlinear open-loop amplifiers and clarify the gain definition for the nonlinear amplifiers. This clarification will help circuit designers identify the non-ideality of close-loop amplifiers.

In section 2, we will model the open-loop operational amplifier with nonlinear components. In section 3, we will discuss the typical close-loop configurations of operational amplifiers with a resistive feedback network. Equivalent gains for different purposes are identified by using the model given in section 2. The conclusion is made in Section 4.

2. Modeling of Operational Amplifiers

For large signal swing and global analysis, all operational amplifiers will work in the nonlinear state. The linear models for amplifiers are simple approximations to the real situation and only work well for small signal swing and local analysis. For low voltage applications, the circuit cannot be accurately described by the small signal model. To have a good performance, signal swing in circuits can no longer be kept "small" compared with the supply voltage. However, it is nearly impossible to model the nonlinearity of amplifiers from the transistor level, since the performance of various architectures of operational amplifiers can be dramatically different from one to another. Their gains and frequency responses are strongly dependent on transistor level designs. This will make the modeling of nonlinear effects prohibitively complicated.

Instead of the bottom-up method from transistor level, we will take a top-down method to model the nonlinear operational amplifier in this paper. Static behavior of any operational amplifier can be described by the DC transfer characteristic in which the input and output could be current or voltage or both. A typical DC transfer characteristic for a differential operational amplifier is shown in Figure 1.

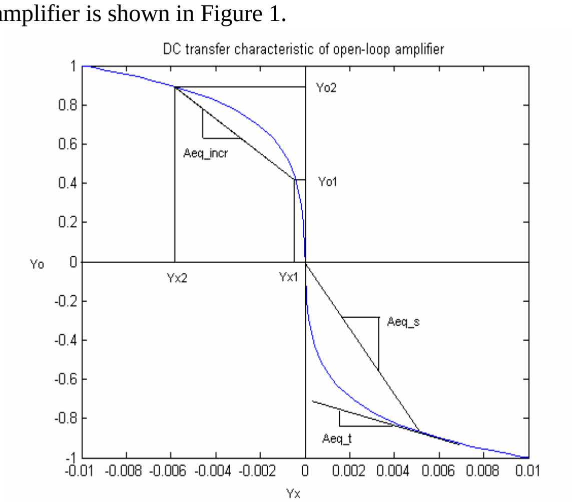

Figure 1 DC transfer characteristic of an open-loop amplifier

* This work is supported, in part, by the National Science Foundation and the Semiconductor Research Corporation.

This DC transfer characteristic curve gives out the input-output relationship of the amplifier

$$Y_o = f(Y_x) \tag{1}$$

where Y_o and Y_x are the output and input of the open-loop amplifier, respectively, and $f(Y_x)$ is a nonlinear function of Y_x.
$f(Y_x)$ will have properties such as

$$f'(Y_x)<0 \tag{2}$$

$$Y_x f''(Y_x)<0 \tag{3}$$

These properties are common to all operational amplifiers. In order to simplify the analysis, voltage amplifier is used in our study but the methodology and all conclusions made here are applicable to other structures as well. In other words, V will be used in the following analysis instead of Y. Meanwhile, most operational amplifiers are used with a feedback network so that a better linearity performance can be achieved. The common knowledge is that higher gain will lead to higher linearity. Unfortunately, with high-gain amplifier-design, the nonlinear effect in the open-loop amplifier will be more serious. That means no matter what kind of structure we use, nonlinear issues in the circuit have to be thoroughly modeled and studied.

For a given amplifier, its performance can include both static and dynamic behaviors. The static performance is characterized by its DC transfer characteristics. Taking an input-output approach, we propose a system model for general open-loop amplifiers, regardless their inner circuit structure, as shown in Figure 2. This model is similar to the commonly used small signal model for a MOS transistor, but it can model the global behavior of any differential operational amplifiers.

Given a DC transfer characteristic curve, we can mathematically include all the non-linearity in g_o while keeping g_m constant. For linear amplifiers, g_o is a constant value. In general situations, g_o can be described as a function of the output voltage, $g_o=g_o(V_o)$. Thus in a static situation,

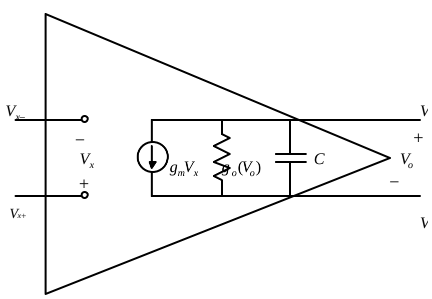

Figure 2 Nonlinear operation amplifier model

$$V_o = -\frac{g_m}{g_o(V_o)} V_x \tag{4}$$

The dynamic performance includes many aspects but we focus on the settling behavior. Although the dynamic performance depends on many things including the filter structure and can be highly complex, a first order approximation is adequate for many applications especially when the amplifier has good phase margin with proper design. Where in such cases, a single capacitor *C* is used to capture the dynamic behavior. C is the combination of the load capacitance and the parasitic capacitance at the output node.

3. Equivalent gains for an open-loop amplifier

The representative application of an open-loop amplifier is the feedback configuration with a resistive network (See Figure 3). This application is the basic application, and study of it will help circuit designers understand the non-ideality of close-loop amplifiers caused by the nonlinearity of open-loop amplifiers. This study can also be applied qualitatively to other situations, such as the switched-capacitor application. In this paper, we will limit our discussion to the feedback configuration with linear resistor network. It is obvious that the external resistor network will change the DC transfer characteristic of the close-loop amplifiers by diverting part of the current. If feedback resistors are large compared with $1/g_0$, such effects can be ignored. In real applications, loading effects do not exist if a buffer is used to isolate the output node or if a capacitive feedback network is used. In this paper, we will make such assumptions so that the feedback network will not affect the DC transfer characteristic of the open-loop amplifier.

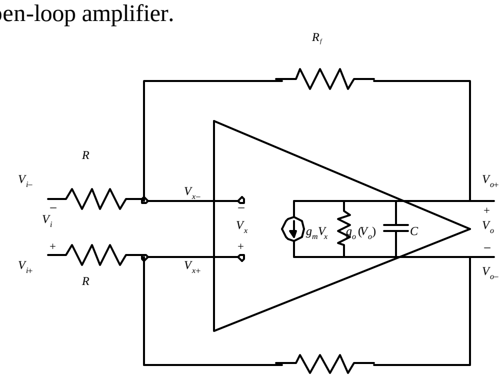

Figure 3 Feedback amplifier with resistive network

For the close-loop amplifier in figure 3, under our assumptions, we can have the following equations:

$$V_o g_o + C\frac{dV_o}{dt} = g_m V_x \tag{5}$$

$$\frac{-V_i + V_x}{R} = \frac{-V_x - V_o}{R_f} \tag{6}$$

The equation that relates the input and output of the close-loop amplifier is

$$\frac{V_i}{R} = \left(\frac{1}{R} + \frac{1}{R_f}\right) \cdot \left(\frac{g_o}{g_m} V_o + \frac{C}{g_m} \cdot \frac{dV_o}{dt}\right) + \frac{1}{R_f} \cdot V_o \tag{7}$$

This equation can be used to solve the output with different initial conditions and different inputs.

For different applications of this feedback configuration, there are different considerations for the behavior of the amplifier. In the following discussion, we will look at four aspects of this configuration.

1) Steady State accuracy of the close-loop circuit

For a large-unit-gain-bandwidth and high-gain amplifier with good phase margin, if the output has enough time to be well settled, what we concern most is the final settling accuracy of the close-loop amplifier.

To evaluate the final settling accuracy, we can look into the steady state step response with zero initial condition. The initial state is that $V_i=V_o=0$. V_i changes to V_1 at $t=0$, and then remains constant. At steady state, we have $V_o = -\frac{g_m}{g_o(V_o)} V_x$. Substituting it into equation (7) and letting $b = \frac{R_f}{R}$, we get the steady step response:

$$V_o = \frac{-b}{1 + (1+b)\frac{g_o(V_o)}{g_m}} V_1 \tag{8}$$

If the open-loop operational amplifier is linear and its gain is A, the steady-state step response will be as follows,

$$V_o = \frac{-b}{1 - (1+b)/A} V_1 \tag{9}$$

Comparing equation (8) with (9), we can define an equivalent open-loop gain for the non-linear amplifier as

$$A_{eq\_s} = -g_m / g_o(V_o) = V_o / V_x \tag{10}$$

As we can expect, this equivalent gain is a nonlinear function of the output voltage. Its physical meaning is the slope of the chord in the DC transfer characteristic curve, from 0 to V_o, marked in Figure 1. To guarantee a required performance, the slope of the largest cord should be used.

2) Settling accuracy with incremental step input

Consider the close-loop circuit shown in Figure 3. The initial state is that at $t=0^-$, $V_i=V_1$ and V_o has already settled to V_{o1}. V_i changes to V_2 at $t=0^+$, and then remains constant. Using the results in 1), we obtain,

At $t=0^-$, $V_i(0^-) = V_1$

$$V_o(0^-) = V_{o1} = \frac{-b}{1 - (1+b)\frac{g_o(V_{o1})}{g_m}} V_1 = \frac{-b}{1 - (1+b)\frac{V_{x1}}{V_{o1}}} V_1 \tag{11}$$

At $t=+$, $dV_o/dt=0$, $V_i(\infty) = V_2$

$$V_o(\infty) = V_{o2} = \frac{-b}{1 + (1+b)\frac{g_o(V_{o2})}{g_m}} V_2 = \frac{-b}{1 - (1+b)\frac{V_{x2}}{V_{o2}}} V_2 \tag{12}$$

where $V_{o1} = f(V_{x1})$, $V_{o2} = f(V_{x2})$ as marked in Figure 1. From (11) and (12), we get

$$V_o(\infty) - V_o(0^-) = \frac{-b}{1 - (1+b)\frac{V_{x2} - V_{x1}}{V_{o2} - V_{o1}}} [V_i(\infty) - V_i(0^-)] \tag{13}$$

For a linear amplifier, we can similarly obtain:

$$V_o(\infty) - V_o(0^-) = \frac{-b}{1 - \frac{(1+b)}{A}} [V_i(\infty) - V_i(0^-)] \tag{14}$$

Comparing (13) to (14), the incremental gain of an open-loop non-linear amplifier can be derived as

$$A_{eq\_incr} = \frac{V_{o2} - V_{o1}}{V_{x2} - V_{x1}} \tag{15}$$

As we can expect, this equivalent gain is initial and final state dependent. Its physical meaning is the slope of the chord in the DC transfer characteristic curve, from V_{o1} to V_{o2}, marked in Figure.1. To guarantee a given increment performance, the cord involving the largest V_o should be used.

3) Settling behavior of the close-loop circuit

Under the condition that the input of the feedback amplifier is piecewise constant (e.g., the output of sample and hold stage in pipelined ADCs), the output always changes before it reaches the final settling value, then it is necessary to look at the settling behavior of the close-loop circuit. From (7), we get

$$\frac{dV_o}{dt} = \underbrace{-\frac{g_m}{(1+b)C} \cdot \left[(1+b)\frac{g_o(V_o)}{g_m} + 1\right] \cdot V_o - \frac{bg_m}{(1+b)C} \cdot V_i}_{= h(V_o)} \tag{16}$$

The settling behavior of equation (16) is basically determined by the derivative of $h(V_o)$ evaluated at the final output voltage.

$$h'(V_o) = -\frac{[g_o(V_o) + V_o g_o'(V_o)]}{C} - \frac{g_m}{C(1+b)} \tag{17}$$

For a linear amplifier with open-loop gain A, the derivative of $h(V_o)$ will be

$$h'(V_o) = \frac{g_m}{C}\frac{1}{A} - \frac{g_m}{C(1+b)} \tag{18}$$

Comparing equation (17) with (18), we can define the settling equivalent open-loop gain for the amplifier as

$$A_{eq\_t} = -g_m \frac{1}{g_o(V_o) + V_o g_o'(V_o)} = \frac{dV_o}{dV_x} \tag{19}$$

This equivalent gain is nothing but the tangent slope of the DC transfer curve at the final output voltage V_o. In real applications, $[g_o(V_o) + V_o g_o'(V_o)] << \frac{g_m}{(1+b)}$ in (17), the settling behavior is strongly dependent on the second term.

In order to compare the settling behavior between a linear amplifier and a nonlinear amplifier, we define $\ln(V_o(t) - V_{of})/(V_o(0) - V_{of})$ as the settling speed. Figure 4 shows MATLAB simulation results of the settling speed difference between a nonlinear amplifier with final tangent slope of 'A' and a linear amplifier with gain 'A'. The difference approaches 0 when the output settles to the final voltage. Thus we verified the settling behavior of a nonlinear amplifier is A_{eq_t} related.

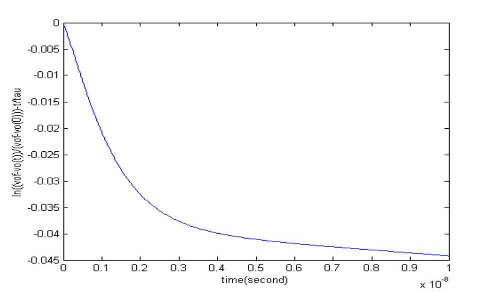

Figure 4 Difference in settling behavior of feedback amplifiers

4) First-order harmonic equivalent gain

Due to the non-linearity of the open-loop amplifier, high-order harmonic distortion may be generated at the output when a pure sinusoidal signal is applied to the input.

From the DC-transfer characteristic we can get

$g_o(V_o) \geq 0$, $g_o(-V_o) = g_o(V_o)$.

Rewrite non-linear differential equation (16) as

$$\frac{dV_o}{dt} = H * V_o + C(V_o)V_o + BV_i \tag{20}$$

where $H = -\frac{g_m}{(1+b)C} < 0$, $B = \frac{-bg_m}{(1+b)C} < 0$,

$C(V_o) = -\frac{g_o(V_o)}{C} \leq 0, \forall V_o$.

With a sinusoidal input $V_i = \sin wt$ (period $T = \frac{2p}{w}$) it can be shown that there exists a unique periodic solution for the differential equation (20) without even-order harmonics [4]

$$V_o(t) = \sum_{k=-\infty}^{\infty} V_{2k+1} e^{j(2k+1)wt} \quad (21)$$

Based on the assumption that the open-loop gain is sufficiently large, we can assume the first-order harmonic dominates the output signal. Then a first-order harmonic equivalent gain can be derived.

Using the describing function method [5], let

$$V_o(t) = Y_1 \sin(wt + q), V_1 = Y_1 e^{jq} \quad (22)$$

We will get

$$Y_1 = \frac{1}{jw + \frac{g_m}{(1+b)C} - \Psi(Y_1)} * \frac{B}{2j} e^{-jq} \quad (23)$$

where $\Psi(Y_1)$ is the describing function of the non-linearity $C(V_o)V_o$.

In case of a linear amplifier,

$$Y_1 = \frac{1}{jw + \frac{g_m}{(1+b)C} - \frac{g_m}{AC}} * \frac{B}{2j} e^{-jq} \quad (24)$$

Comparing equation (23) to (24), a first-order harmonic equivalent gain can be derived as

$$A_{eq} = \frac{g_m}{C\Psi(Y_1)} \quad (25)$$

This equivalent gain is very close to steady state equivalent gain.

5) Distortion to sinusoidal input

Based on the assumption that the first-order harmonic dominates the output signal and from the equation (21), we know the third-order harmonic distortion will dictate the SFDR performance of the feedback amplifier.

Under such conditions, the third-order harmonic is given by

$$V_3 = \frac{1}{T} \int_0^T \Psi(Y_1 \sin(wt + q)) e^{-j3wt} dt / (j3w + H) \quad (26)$$

SFDR can be calculated with V_3 and Y_1. However, it is extremely difficult to get explicit expression of V_3 due to the complexity of the nonlinear function.

If the input sinusoidal signal's period is much longer than the time constant of the feedback amplifier, we do not need to concern about the issue of settling within finite settling time. However, high-order harmonic distortion will lead to output non-linearity. In some applications, high linearity is very important, such as ADC design, so the distortion introduced by the nonlinearity of amplifiers need to be estimated so that designers can make wise decisions on how to select the circuit parameter. A new term linearity index (LI) is introduced to do the estimation:

$$LI = \frac{A_{eq\_s}(0) A_{eq\_s}(V_{om})}{A_{eq\_s}(0) - A_{eq\_s}(V_{om})} \quad (27)$$

With the linearity index, we can estimate THD (total harmonic distortion) of the output signal using the following approximate equation (28). Assuming all other circuits are linear except the open-loop amplifier,

$$THD \le -20 \log_{10} LI - 12 \quad (dB) \quad (28)$$

Simulation results show that this equation gives good estimation with very small error. The comparison of estimated results and simulated results are shown in Table 1, which summarizes the results for two categories of open-loop amplifiers: low gain amplifiers with good linearity and high gain amplifiers with weak linearity. For all the results in the table, V_{om}=1 and â=1. From Table 1 we can see that simulation results fit the estimation results very well. The estimation method by using linearity index is quite effective.

Table 1 Estimated THD and simulated THD for close-loop amplifiers' output

$A_{eq\_s}(0)$	$A_{eq\_s}(1)$	LI	THD(est.) (dB)	THD(sim.) (dB)
10	9.99	10000	-92	-92.3
100	90.91	1000	-72	-72.31
100	98.04	5,000	-85.98	-86.3
100000	991	1001	-72.009	-72.0566
1000000	999	1000	-72	-72.056
100000	9,091	10000	-92	-92.043

4. Conclusion

Because of the intrinsic non-ideality of semiconductor devices, low voltage high gain open-loop amplifiers exhibit high nonlinearities. It always turns out to be a confusing problem for circuit designers to make gain measurement and understand the non-ideality of close-loop configuration due to the nonlinearity of open-loop amplifiers. To solve this problem, simple but effective model for nonlinear open-loop amplifiers is given, and the small signal gain and large signal gain are defined. Based on this model and the gain definitions, the non-linearities of feedback amplifiers with a resistor feedback network for different considerations are inspected. Three equivalent gains are defined for different cases. An effective THD estimation method is given based on the definition of linearity index. It is very obvious that clarified gain definitions for non-linear open-loop amplifiers help a lot to identify the non-idealities of feedback amplifiers.

References

[1] David A. Johns and Ken Martin, "Chapter 5: Basic Opamp Design and Compensation", "Analog Integrated Circuit Design", John Wiley & Sons, Inc., 1997

[2] Phillip E. Allen and Douglas R. Holberg, "Chapter 9: Switched Capacitor Circuits", "CMOS Analog Circuit Design", Second Edition, Oxford University Press, 1997

[3] Yonghui Tang and Randall Geiger, "Effects of Open-loop Nonlinearity on Linearity of Feedback Amplifiers", in Proc. IEEE ISCAS 2002, May 2002

[4] Ju. L. Dalechi and M. G. Kre n, "Stability of Solutions of Differential Equations in Banach Space", Technical Report, Department of Electrical and Computer Engineering, Iowa State University, 2002

[5] Hassan K. Khall, "Nonlinear Systems", Second Edition, Prentice-Hall Inc., 1996