2.1 Let \( C \subseteq \mathbb{R}^n \) be a convex set, with \( x_1, \ldots, x_k \in C \), and let \( \theta_1, \ldots, \theta_k \in \mathbb{R} \) satisfy \( \theta_i \geq 0 \), \( \theta_1 + \cdots + \theta_k = 1 \). Show that \( \theta_1 x_1 + \cdots + \theta_k x_k \in C \). (The definition of convexity is that this holds for \( k = 2 \); you must show it for arbitrary \( k \).) Hint. Use induction on \( k \).

**Solution.** This is readily shown by induction from the definition of convex set. We illustrate the idea for \( k = 3 \), leaving the general case to the reader. Suppose that \( x_1, x_2, x_3 \in C \), and \( \theta_1 + \theta_2 + \theta_3 = 1 \) with \( \theta_1, \theta_2, \theta_3 \geq 0 \). We will show that \( y = \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \in C \). At least one of the \( \theta_i \) is not equal to one; without loss of generality we can assume that \( \theta_1 \neq 1 \). Then we can write

\[
y = \theta_1 x_1 + (1 - \theta_1)(\mu_2 x_2 + \mu_3 x_3)
\]

where \( \mu_2 = \theta_2/(1 - \theta_1) \) and \( \mu_2 = \theta_3/(1 - \theta_1) \). Note that \( \mu_2, \mu_3 \geq 0 \) and

\[
\mu_1 + \mu_2 = \frac{\theta_2 + \theta_3}{1 - \theta_1} = \frac{1 - \theta_1}{1 - \theta_1} = 1.
\]

Since \( C \) is convex and \( x_2, x_3 \in C \), we conclude that \( \mu_2 x_2 + \mu_3 x_3 \in C \). Since this point and \( x_1 \) are in \( C \), \( y \in C \).

2.2 Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

**Solution.** We prove the first part. The intersection of two convex sets is convex. Therefore if \( S \) is a convex set, the intersection of \( S \) with a line is convex.

Conversely, suppose the intersection of \( S \) with any line is convex. Take any two distinct points \( x_1 \) and \( x_2 \in S \). The intersection of \( S \) with the line through \( x_1 \) and \( x_2 \) is convex. Therefore convex combinations of \( x_1 \) and \( x_2 \) belong to the intersection, hence also to \( S \).
2.5 What is the distance between two parallel hyperplanes \( \{ x \in \mathbb{R}^n \mid a^T x = b_1 \} \) and \( \{ x \in \mathbb{R}^n \mid a^T x = b_2 \} \)?

**Solution.** The distance between the two hyperplanes is \( |b_1 - b_2|/\|a\|_2 \). To see this, consider the construction in the figure below.

The distance between the two hyperplanes is also the distance between the two points \( x_1 \) and \( x_2 \) where the hyperplane intersects the line through the origin and parallel to the normal vector \( a \). These points are given by

\[
\begin{align*}
x_1 &= (b_1/\|a\|_2^2) a, \\
x_2 &= (b_2/\|a\|_2^2) a,
\end{align*}
\]

and the distance is

\[
\|x_1 - x_2\|_2 = |b_1 - b_2|/\|a\|_2.
\]
2.8 Which of the following sets $S$ are polyhedra? If possible, express $S$ in the form $S = \{ x \mid Ax \preceq b, Fx = g \}$.

(a) $S = \{ y_1 a_1 + y_2 a_2 \mid -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1 \}$, where $a_1, a_2 \in \mathbb{R}^n$.

(b) $S = \{ x \in \mathbb{R}^n \mid x \succeq 0, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2 \}$, where $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

(c) $S = \{ x \in \mathbb{R}^n \mid x \succeq 0, x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1 \}$.

(d) $S = \{ x \in \mathbb{R}^n \mid x \succeq 0, x^T y \leq 1 \text{ for all } y \text{ with } \sum_{i=1}^n |y_i| = 1 \}$.

Solution.

(a) $S$ is a polyhedron. It is the parallelogram with corners $a_1 + a_2, a_1 - a_2, -a_1 + a_2, -a_1 - a_2$, as shown below for an example in $\mathbb{R}^2$.

![Parallelogram diagram]

For simplicity we assume that $a_1$ and $a_2$ are independent. We can express $S$ as the intersection of three sets:

- $S_1$: the plane defined by $a_1$ and $a_2$
- $S_2 = \{ z + y_1 a_1 + y_2 a_2 \mid a_1^T z = a_2^T z = 0, -1 \leq y_1 \leq 1 \}$. This is a slab parallel to $a_2$ and orthogonal to $S_1$
- $S_3 = \{ z + y_1 a_1 + y_2 a_2 \mid a_1^T z = a_2^T z = 0, -1 \leq y_2 \leq 1 \}$. This is a slab parallel to $a_1$ and orthogonal to $S_1$

Each of these sets can be described with linear inequalities.

- $S_1$ can be described as $u_k^T x = 0, \ k = 1, \ldots, n - 2$

where $u_k$ are $n - 2$ independent vectors that are orthogonal to $a_1$ and $a_2$ (which form a basis for the nullspace of the matrix $[a_1 \ a_2]^T$).
2 Convex sets

Let \( c_1 \) be a vector in the plane defined by \( a_1 \) and \( a_2 \), and orthogonal to \( a_2 \). For example, we can take

\[
c_1 = a_1 - \frac{a_1^T a_2}{\|a_2\|^2} a_2.
\]

Then \( x \in S_2 \) if and only if

\[
-|c_1^T a_1| \leq c_1^T x \leq |c_1^T a_1|.
\]

Similarly, let \( c_2 \) be a vector in the plane defined by \( a_1 \) and \( a_2 \), and orthogonal to \( a_1 \), e.g.,

\[
c_2 = a_2 - \frac{a_2^T a_1}{\|a_1\|^2} a_1.
\]

Then \( x \in S_3 \) if and only if

\[
-|c_2^T a_2| \leq c_2^T x \leq |c_2^T a_2|.
\]

Putting it all together, we can describe \( S \) as the solution set of 2\( n \) linear inequalities

\[
v_k^T x \leq 0, \quad k = 1, \ldots, n - 2
\]

\[
-v_k^T x \leq 0, \quad k = 1, \ldots, n - 2
\]

\[
e_1^T x \leq |e_1^T a_1|
\]

\[
-e_1^T x \leq |e_1^T a_1|
\]

\[
e_2^T x \leq |e_2^T a_2|
\]

\[
-e_2^T x \leq |e_2^T a_2|
\]

(b) \( S \) is a polyhedron, defined by linear inequalities \( x_k \geq 0 \) and three equality constraints.

(c) \( S \) is not a polyhedron. It is the intersection of the unit ball \( \{x \mid \|x\|_2 \leq 1\} \) and the nonnegative orthant \( \mathbb{R}^n_+ \). This follows from the following fact, which follows from the Cauchy-Schwarz inequality:

\[
x^T y \leq 1 \text{ for all } y \text{ with } \|y\|_2 = 1 \iff \|x\|_2 \leq 1.
\]

Although in this example we define \( S \) as an intersection of halfspaces, it is not a polyhedron, because the definition requires infinitely many halfspaces.

(d) \( S \) is a polyhedron. \( S \) is the intersection of the set \( \{x \mid |x_k| \leq 1, \quad k = 1, \ldots, n\} \) and the nonnegative orthant \( \mathbb{R}^n_+ \). This follows from the following fact:

\[
x^T y \leq 1 \text{ for all } y \text{ with } \sum_{i=1}^n |y_i| = 1 \iff |x_i| \leq 1, \quad i = 1, \ldots, n.
\]

We can prove this as follows. First suppose that \( |x_i| \leq 1 \) for all \( i \). Then

\[
x^T y = \sum_i x_i y_i \leq \sum_i |x_i| |y_i| \leq \sum_i |y_i| = 1
\]

if \( \sum_i |y_i| = 1 \).

Conversely, suppose that \( x \) is a nonzero vector that satisfies \( x^T y \leq 1 \) for all \( y \) with \( \sum_i |y_i| = 1 \). In particular we can make the following choice for \( y \): let \( k \) be an index for which \( |x_k| = \max_i |x_i| \), and take \( y_k = 1 \) if \( x_k > 0 \), \( y_k = -1 \) if \( x_k < 0 \), and \( y_i = 0 \) for \( i \neq k \). With this choice of \( y \) we have

\[
x^T y = \sum_i x_i y_i = y_k x_k = |x_k| = \max_i |x_i|.
\]
2.10 Solution set of a quadratic inequality. Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \{ x \in \mathbb{R}^n \mid x^T Ax + b^T x + c \leq 0 \},$$

with $A \in \mathbb{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

(a) Show that $C$ is convex if $A \succeq 0$.

(b) Show that the intersection of $C$ and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda gg^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

Are the converses of these statements true?

**Solution.** A set is convex if and only if its intersection with an arbitrary line $\{ \tilde{x} + tv \mid t \in \mathbb{R} \}$ is convex.

(a) We have

$$(\tilde{x} + tv)^T A(\tilde{x} + tv) + b^T (\tilde{x} + tv) + c = \alpha t^2 + \beta t + \gamma$$

where

$$\alpha = v^T Av, \quad \beta = b^T v + 2\tilde{x}^T Av, \quad \gamma = c + b^T \tilde{x} + \tilde{x}^T A \tilde{x}.$$
2.12 Which of the following sets are convex?

(a) A slab, i.e., a set of the form \( \{ x \in \mathbb{R}^n \mid a \leq a^T x \leq \beta \} \).

(b) A rectangle, i.e., a set of the form \( \{ x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n \} \). A rectangle is sometimes called a hyperrectangle when \( n > 2 \).
(c) A *wedge*, i.e., \( \{ x \in \mathbb{R}^n \mid a_1^T x \leq b_1, \ a_2^T x \leq b_2 \} \).

(d) The set of points closer to a given point than a given set, i.e.,
\[
\{ x \mid \| x - x_0 \|_2 \leq \| x - y \|_2 \text{ for all } y \in S \}
\]
where \( S \subseteq \mathbb{R}^n \).

(e) The set of points closer to one set than another, i.e.,
\[
\{ x \mid \text{dist}(x, S) \leq \text{dist}(x, T) \},
\]
where \( S, T \subseteq \mathbb{R}^n \), and

\[
\text{dist}(x, S) = \inf \{ \| x - z \|_2 \mid z \in S \}.
\]

(f) [HUL93, volume 1, page 93] The set
\[
\{ x \mid x + S_2 \subseteq S_1 \}
\]
where \( S_1, S_2 \subseteq \mathbb{R}^n \) with \( S_1 \) convex.

(g) The set of points whose distance to \( a \) does not exceed a fixed fraction \( \theta \) of the distance to \( b \), i.e., the set
\[
\{ x \mid \| x - a \|_2 \leq \theta \| x - b \|_2 \}
\]
you can assume \( a \neq b \) and \( 0 \leq \theta \leq 1 \).

**Solution.**

(a) A slab is an intersection of two halfspaces, hence it is a convex set (and a polyhedron).

(b) As in part (a), a rectangle is a convex set and a polyhedron because it is a finite intersection of halfspaces.

(c) A wedge is an intersection of two halfspaces, so it is convex set. It is also a polyhedron. It is a cone if \( b_1 = 0 \) and \( b_2 = 0 \).

(d) This set is convex because it can be expressed as
\[
\bigcap_{y \in S} \{ x \mid \| x - x_0 \|_2 \leq \| x - y \|_2 \},
\]
i.e., an intersection of halfspaces. (For fixed \( y \), the set
\[
\{ x \mid \| x - x_0 \|_2 \leq \| x - y \|_2 \}
\]
is a halfspace; see exercise 2.9).

(e) In general this set is not convex, as the following example in \( \mathbb{R} \) shows. With \( S = \{-1, 1\} \) and \( T = \{0\} \), we have
\[
\{ x \mid \text{dist}(x, S) \leq \text{dist}(x, T) \} = \{ x \in \mathbb{R} \mid x \leq -1/2 \text{ or } x \geq 1/2 \}
\]
which clearly is not convex.

(f) This set is convex. \( x + S_2 \subseteq S_1 \) if \( x + y \in S_1 \) for all \( y \in S_2 \). Therefore
\[
\{ x \mid x + S_2 \subseteq S_1 \} = \bigcap_{y \in S_2} \{ x \mid x + y \in S_1 \} = \bigcap_{y \in S_2} (S_1 - y),
\]
the intersection of convex sets \( S_1 - y \).

(g) The set is convex, in fact a ball.
\[
\begin{align*}
\{ x \mid \| x - a \|_2 \leq \theta \| x - b \|_2 \} \\
= \{ x \mid \| x - a \|_2^2 \leq \theta^2 \| x - b \|_2^2 \} \\
= \{ x \mid (1 - \theta^2) x^T x - 2(\theta^2 b^T x) + (\theta^2 a^T a - \theta^2 b^T b) \leq 0 \}
\end{align*}
\]